REGULARIZATION OF POINT VORTICES FOR THE EULER EQUATION IN DIMENSION TWO, PART II

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Abstract. In this paper, we continue to construct stationary classical solutions of the incompressible Euler equation approximating singular stationary solutions of this equation. This procedure now is carried out by constructing solutions to the following elliptic problem

$$\begin{cases} -\varepsilon^2\Delta u = \sum_{i=1}^m \chi_{\Omega_i^+}\big(u - q - \frac{\kappa_i^+}{2\pi}\ln\frac{1}{\varepsilon}\big)_+^p - \sum_{j=1}^n \chi_{\Omega_j^-}\big(q - \frac{\kappa_j^-}{2\pi}\ln\frac{1}{\varepsilon} - u\big)_+^p, & x\in\Omega,\\ u=0, & x\in\partial\Omega, \end{cases}$$

where $p>1$, $\Omega\subset\mathbb{R}^2$ is a simply connected bounded domain, Ω_i^+ and Ω_j^- are mutually disjoint subdomains of Ω, q is a harmonic function.

We showed that if Ω is a simply-connected smooth domain, then for any given C^1-stable critical point of Kirchhoff-Routh function $\mathcal{W}(x_1^+,\cdots,x_m^+,x_1^-,\cdots,x_n^-)$ with $\kappa_i^+>0\,(i=1,\cdots,m)$ and $\kappa_j^->0\,(j=1,\cdots,n)$, then there is a stationary classical solution approximating stationary $m+n$ points vortex solution of incompressible Euler equations with total vorticity $\sum_{i=1}^m\kappa_i - \sum_{j=1}^n\kappa_j^-$.

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1. Introduction and main results

The incompressible Euler equations

$$\begin{cases} \mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P,\\ \nabla\cdot\mathbf{v} = 0, \end{cases} \tag{1.1}$$

describe the evolution of the velocity $\mathbf{v}$ and the pressure P in an incompressible flow. In $\mathbb{R}^2$, the vorticity of the flow is defined by $\omega = \nabla\times\mathbf{v} := \partial_1 v_2 - \partial_2 v_1$, which satisfies the equation

$$\omega_t + \mathbf{v}\cdot\nabla\omega = 0.$$

Suppose that ω is known, then the velocity $\mathbf{v}$ can be recovered by Biot-Savart law as following:

$$\mathbf{v} = \omega * \frac{1}{2\pi}\frac{-x^\perp}{|x|^2},$$

where $x^\perp = (x_2, -x_1)$ if $x = (x_1, x_2)$. One special singular solutions of Euler equations is given by $\omega = \sum_{i=1}^m \kappa_i \delta_{x_i(t)}$, which is related

$$\mathbf{v} = -\sum_{i=1}^m \frac{\kappa_i}{2\pi} \frac{(x - x_i(t))^\perp}{|x - x_i(t)|^2}.$$

and the positions of the vortices $x_i : \mathbb{R} \to \mathbb{R}^2$ satisfy the following Kirchhoff law:

$$\kappa_i \frac{dx_i}{dt} = (\nabla_{x_i} \mathcal{W})^\perp$$

where $\mathcal{W}$ is the so called Kirchhoff-Routh function defined by

$$\mathcal{W}(x_1, \cdots, x_m) = \frac{1}{2} \sum_{i \neq j}^m \frac{\kappa_i \kappa_j}{2\pi} \log \frac{1}{|x_i - x_j|}.$$

In simply-connected bounded domain $\Omega \subset \mathbb{R}^2$, similar singular solutions also exist. Suppose that the normal component of $\mathbf{v}$ vanishes on $\partial\Omega$, then the Kirchhoff-Routh function is

$$\mathcal{W}(x_1, \cdots, x_m) = \frac{1}{2} \sum_{i \neq j}^m \kappa_i \kappa_j G(x_i, x_j) + \frac{1}{2} \sum_{i=1}^m \kappa_i^2 H(x_i, x_i), \tag{1.2}$$

where G is the Green function of $-\Delta$ on Ω with 0 Dirichlet boundary condition and H is its regular part (the Robin function).

Let v_n be the outward component of the velocity $\mathbf{v}$ on the boundary $\partial\Omega$, then we see that $\int_{\partial\Omega} v_n = 0$ due to the fact that $\nabla \cdot \mathbf{v} = 0$. Suppose that $\mathbf{v}_0$ is the unique harmonic field whose normal component on the boundary $\partial\Omega$ is v_n, then $\mathbf{v}_0$ satisfies

$$\begin{cases} \nabla \cdot \mathbf{v}_0 = 0, \text{ in } \Omega, \\ \nabla \times \mathbf{v}_0 = 0, \text{ in } \Omega, \\ n \cdot \mathbf{v}_0 = v_n, \text{ on } \partial\Omega. \end{cases} \tag{1.3}$$

If Ω is simply-connected, then $\mathbf{v}_0$ can be written $\mathbf{v}_0 = (\nabla \psi_0)^\perp$, where the stream function ψ_0 is determined up to a constant by

$$\begin{cases} -\Delta \psi_0 = 0, \text{ in } \Omega, \\ -\dfrac{\partial \psi_0}{\partial \tau} = v_n, \text{ on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\frac{\partial \psi_0}{\partial \tau}$ denotes the tangential derivative on $\partial\Omega$. The Kirchhoff-Routh function associated to the vortex dynamics becomes(see Lin [24])

$$\mathcal{W}(x_1, \cdots, x_m) = \frac{1}{2} \sum_{i \neq j}^m \kappa_i \kappa_j G(x_i, x_j) + \frac{1}{2} \sum_{i=1}^m \kappa_i^2 H(x_i, x_i) + \sum_{i=1}^m \kappa_i \psi_0(x_i). \tag{1.5}$$

For m clockwise vortices motion (corresponding to $\kappa_i^+ > 0$) and n anti-clockwise vortices motion (corresponding to $-\kappa_j^- < 0$), the Kirchhoff-Routh function associated to the vortex dynamics becomes

$$\begin{aligned}\mathcal{W}(x_1^+, \cdots, x_m^+, x_1^-, \cdots, x_n^-) =& \frac{1}{2}\sum_{i,k=1, i\neq k}^{m} \kappa_i^+\kappa_k^+ G(x_i^+, x_k^+) + \frac{1}{2}\sum_{j,l=1, j\neq l}^{n} \kappa_j^-\kappa_l^- G(x_j^-, x_l^-) \\ &+ \frac{1}{2}\sum_{i=1}^{m}(\kappa_i^+)^2 H(x_i^+, x_i^+) + \frac{1}{2}\sum_{j=1}^{n}(\kappa_j^-)^2 H(x_j^-, x_j^-) \\ &- \sum_{i=1}^{m}\sum_{j=1}^{n}\kappa_i^+\kappa_j^- G(x_i^+, x_j^-) + \sum_{i=1}^{m}\kappa_i^+\psi_0(x_i^+) - \sum_{j=1}^{n}\kappa_j^-\psi_0(x_j^-).\end{aligned} \tag{1.6}$$

It is known that critical points of the Kirchhoff-Routh function $\mathcal{W}$ give rise to stationary vortex points solutions of the Euler equations. As for the existence of critical points of $\mathcal{W}$ given by (1.2), we refer to [5].

Roughly speaking, there are two methods to construct stationary solutions of the Euler equation, which are the vorticity method and the stream-function method. The vorticity method was first established by Arnold and Khesin [3] and further developed by Burton [7] and Turkington [32].

The stream-function method consists in observing that if ψ satisfies $-\Delta\psi = f(\psi)$ for some function $f \in C^1(\mathbb{R})$, then $\mathbf{v} = (\nabla\psi)^\perp$ and $P = F(\psi) - \frac{1}{2}|\nabla\psi|^2$ is a stationary solution to the Euler equations, where $(\nabla\psi)^\perp := (\frac{\partial\psi}{\partial x_2}, -\frac{\partial\psi}{\partial x_1}), F(t) = \int_0^t f(s)ds$. Moreover, the velocity $\mathbf{v}$ is irrotational on the set where $f(\psi) = 0$.

Set $q = -\psi_0$ and $u = \psi - \psi_0$, then u satisfies the following boundary value problem

$$\begin{cases}-\Delta u = f(u-q), & x \in \Omega, \\ u = 0, & x \in \partial\Omega.\end{cases} \tag{1.7}$$

In addition, if we suppose that $\inf_\Omega q > 0$ and $f(t) = 0,\ t \leq 0$, the vorticity set $\{x : f(\psi) > 0\}$ is bounded away from the boundary.

The motivation to study (1.7) is to justify the weak formulation for point vortex solutions of the incompressible Euler equations by approximating these solutions with classical solutions.

Marchioro and Pulvirenti [26] have approximated these solutions on finite time intervals by considering regularized initial data for the vorticity. On the other hand, the stationary point vortex solutions can also be approximated by stationary classical solutions. See e.g. [1, 2, 4, 6, 19, 28, 29, 31, 32, 33, 34] and the references therein.

In [18] Elcrat and Miller, by a rearrangements of functions, have studied steady, inviscid flows in two dimensions which have concentrated regions of vorticity. In particular, they studied such flows which "desingularize" a configuration of point vortices in stable equilibrium with an irrotational flow, which generalized their earlier work for one vortex [16][17] which in turn were based on results of Turkington [32]. As pointed by Elcrat and Miller, an

essential hypothesis in their existence proof was that the vorticity was in a neighborhood of a stable point vortex configuration. Saffman and Sheffield [30] have found an example of a steady flow in aerodynamics with a single point vortex which is stable for a certain range of the parameters. This has been generalized in [16], where some examples computationally of stable configurations of two point vortices were briefly discussed. Further examples of multiple point vortex configurations are given in [27], where a theorem on the existence of such configurations is also given.

It is worth pointing out that except [18] the above approximations can just give explanation for the formulation to single point vortex solutions. D. Smets and J. Van Schaftingen [31] investigated the following problem

$$\begin{cases} -\varepsilon^2\Delta u = \left(u - q - \frac{\kappa}{2\pi}\ln\frac{1}{\varepsilon}\right)_+^p, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

and gave the exact asymptotic behavior and expansion of the least energy solution by estimating the upper bounds on the energy. The solutions for (1.8) in [31] were obtained by finding a minimizer of the corresponding functional in a suitable function space, which can only give approximation to a single point non-vanishing vortex. In [13], we have shown that multi-point vortex solutions can be approximated by stationary classical solutions.

Concerning regularization of pairs of vortices, D. Smets and J. Van Schaftingen [31] also studied the following problem

$$\begin{cases} -\varepsilon^2\Delta u = \left(u - q - \frac{\kappa}{2\pi}\ln\frac{1}{\varepsilon}\right)_+^p - \left(q - \frac{\kappa}{2\pi}\ln\frac{1}{\varepsilon} - u\right)_+^p, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

and obtained the exact asymptotic behavior and expansion of the least energy solution by similar methods for (1.8). This method is hard to obtain multiple non-vanishing pairs of vortices solutions.

In this paper, we approximate stationary vortex solutions of Euler equations (1.1) with multiple non-vanishing pairs of vortices solutions by stationary classical solutions. Our main result concerning (1.1) is the following:

Theorem 1.1. *Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded simply-connected smooth domain. Let $v_n : \partial\Omega \to \mathbb{R}$ be such that $v_n \in L^s(\partial\Omega)$ for some $s > 1$ satisfying $\int_{\partial\Omega} v_n = 0$. Let $\kappa_i^+ > 0, \kappa_j^- > 0$, $i = 1, \cdots, m$, $j = 1, \ldots, n$. Then, for any given C^1-stable critical point $(x_{1,*}^+, \cdots, x_{m,*}^+, x_{1,*}^-, \cdots, x_{n,*}^-)$ of Kirchhoff-Routh function $\mathcal{W}$ defined by (1.6), there exists $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0)$, problem (1.1) has a stationary solution $\mathbf{v}_\varepsilon$ with outward boundary flux given by v_n, such that its vorticities $\omega_\varepsilon^\pm$ satisfying*

$$supp(\omega_\varepsilon^+) \subset \cup_{i=1}^m B(x_{i,\varepsilon}, C\varepsilon) \text{ for } x_{i,\varepsilon}^+ \in \Omega,\ i = 1, \cdots, m,$$

$$supp(\omega_\varepsilon^-) \subset \cup_{j=1}^n B(x_{j,\varepsilon}^-, C\varepsilon) \text{ for } x_{j,\varepsilon}^- \in \Omega,\ j = 1, \cdots, n$$

and as $\varepsilon \to 0$

$$\int_\Omega \omega_\varepsilon \to \sum_{i=1}^m \kappa_i^+ - \sum_{j=1}^n \kappa_j^-,$$

$$(x_{1,\varepsilon}^+,\cdots,x_{m,\varepsilon}^+,x_{1,\varepsilon}^-,\cdots,x_{n,\varepsilon}^-)\to(x_{1,*}^+,\cdots,x_{m,*}^+,x_{1,*}^-,\cdots,x_{n,*}^-).$$

Remark 1.2. The simplest case, corresponding to pairs of vortices ($m=n=1$) was studied by Smets and Van Schaftingen [31] by minimizing the corresponding energy functional. In their paper as $\varepsilon\to 0$, $\mathcal{W}(x_{1,\varepsilon}^+,x_{1,\varepsilon}^-)\to\sup\limits_{x_1^+,x_1^-\in\Omega,\,x_1^+\neq x_1^-}\mathcal{W}(x_1^+,x_1^-)$. Even in the case $m=n=1$, our result extends theirs to general critical points (with additional assumption that the critical point is non-degenerate or stable in the sense of C^1). The method used in [31] can not be applied to deal with general critical point cases. The method used here is constructive and is completely different from theirs.

Remark 1.3. In this case that $m=n=1$ suppose that $(x_{1,*}^+,\,x_{1,*}^-)$ is a strict local maximum(or minimum) point of Kirchhoff-Routh function $\mathcal{W}(x^+,\,x^-)$ defined by (1.6), statement of Theorem 1.1 still holds which can be proved similarly (see Remark 1.5). Thus we can obtain corresponding existence result in [31].

Theorem 1.1 is proved via considering the following problem

$$\begin{cases}-\varepsilon^2\Delta u=\sum_{i=1}^m\chi_{\Omega_i^+}\big(u-q-\frac{\kappa_i^+}{2\pi}\ln\frac{1}{\varepsilon}\big)_+^p-\sum_{j=1}^n\chi_{\Omega_j^-}\big(q-\frac{\kappa_j^-}{2\pi}\ln\frac{1}{\varepsilon}-u\big)_+^p, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}\tag{1.10}$$

where $p>1$, $q\in C^2(\Omega)$, $\Omega\subset\mathbb{R}^2$ is a bounded domain, $\Omega_i^+(i=1\cdots,m)$ and $\Omega_j^-(j=1\cdots,n)$ are mutually disjoint subdomains of Ω such that $x_{i,*}^+\in\Omega_i^+$, and $x_{j,*}^-\in\Omega_j^-$.

Theorem 1.4. *Suppose $q\in C^2(\Omega)$. Then for any given $\kappa_i^+>0,\kappa_j^->0$, $i=1,\cdots,m$, $j=1,\dots,n$ and for any given C^1-stable critical point $(x_{1,*}^+,\cdots,x_{m,*}^+,x_{1,*}^-,\cdots,x_{n,*}^-)$ of Kirchhoff-Routh function $\mathcal{W}$ defined by (1.6), there exists $\varepsilon_0>0$, such that for each $\varepsilon\in(0,\varepsilon_0)$, (1.10) has a solution u_ε, such that the set $\Omega_{\varepsilon,i}^+=\{x:u_\varepsilon(x)-\frac{\kappa_i^+}{2\pi}\ln\frac{1}{\varepsilon}-q(x)>0\}\subset\subset\Omega_i^+$, $i=1,\cdots,m$, $\Omega_{\varepsilon,j}^-=\{x:u_\varepsilon(x)-\frac{\kappa_j^-}{2\pi}\ln\frac{1}{\varepsilon}-q(x)>0\}\subset\subset\Omega_j^-$, $j=1,\cdots,n$ and as $\varepsilon\to0$, each $\Omega_{\varepsilon,i}^\pm$ shrinks to $x_{i,*}^\pm\in\Omega$.*

Remark 1.5. For the case $m=n=1$, suppose that $(x_{1,*}^+,\,x_{1,*}^-)$ is a strict local maximum(or minimum) point of Kirchhoff-Routh function $\mathcal{W}(x)$ defined by (1.6), then statement of Theorem 1.4 still holds. This conclusion can be proved by making corresponding modification of the proof of Theorem 1.4 in obtaining critical point of $K(z)$ defined by (4.1)(see Propositions 2.3,2.5 and 2.6 in [12] for detailed arguments).

As in [13], we prove Theorem 1.4 by considering an equivalent problem of (1.10) instead. Let $w=\frac{2\pi}{|\ln\varepsilon|}u$ and $\delta=\varepsilon(\frac{2\pi}{|\ln\varepsilon|})^{\frac{p-1}{2}}$, then (1.10) becomes

$$\begin{cases}-\delta^2\Delta w=\sum_{i=1}^m\chi_{\Omega_i^+}\Big(w-\kappa_i^+-\frac{2\pi}{|\ln\varepsilon|}q(x)\Big)_+^p-\sum_{j=1}^m\chi_{\Omega_j^-}\Big(\frac{2\pi}{|\ln\varepsilon|}q(x)-\kappa_j^--w\Big)_+^p, & \text{in }\Omega,\\ w=0, & \text{on }\partial\Omega.\end{cases}\tag{1.11}$$

We will use a reduction argument to prove Theorem 1.4. To this end, we need to construct an approximate solution for (1.11). For the problem studied in this paper, the corresponding "limit" problem in $\mathbb{R}^2$ has no bounded nontrivial solution. So, we will follow the method in [14, 15] to construct an approximate solution. Since there are two parameters δ, ε in problem (1.11) and two terms in nonlinearity, which causes some difficulty, we must take this influence into careful consideration and give delicate estimates in order to perform the reduction argument. For example we need to consider $(s^+_{1,\delta},\cdots,s^+_{m,\delta},s^-_{1,\delta},\cdots,s^-_{n,\delta})$ and $(a^+_{1,\delta},\cdots,a^+_{m,\delta},a^-_{1,\delta},\cdots,a^-_{n,\delta})$ together in Lemma 2.1.

As a final remark, we point out that problem (1.11) can be considered as a free boundary problem. Similar problems have been studied extensively. The reader can refer to [11, 13, 14, 15, 20, 23] for more results on this kind of problems.

This paper is organized as follows. In section 2, we construct the approximate solution for (1.11). We will carry out a reduction argument in section 3 and the main results will be proved in section 4. We put some basic estimates used in sections 3 and 4 in the appendix.

2. Approximate solutions

In the section, we will construct approximate solutions for (1.11).

Let $R>0$ be a large constant, such that for any $x\in\Omega$, $\Omega\subset\subset B_R(x)$. Consider the following problem:

$$\begin{cases}-\delta^2\Delta w=(w-a)^p_+, & \text{in } B_R(0),\\ w=0, & \text{on } \partial B_R(0),\end{cases}\tag{2.1}$$

where $a>0$ is a constant. Then, (2.1) has a unique solution $W_{\delta,a}$, which can be written as

$$W_{\delta,a}(x)=\begin{cases}a+\delta^{2/(p-1)}s_\delta^{-2/(p-1)}\phi\big(\frac{|x|}{s_\delta}\big), & |x|\le s_\delta,\\ a\ln\frac{|x|}{R}/\ln\frac{s_\delta}{R}, & s_\delta\le|x|\le R,\end{cases}\tag{2.2}$$

where $\phi(x)=\phi(|x|)$ is the unique solution of

$$-\Delta\phi=\phi^p,\quad \phi>0,\ \phi\in H^1_0\big(B_1(0)\big)$$

and $s_\delta\in(0,R)$ satisfies

$$\delta^{2/(p-1)}s_\delta^{-2/(p-1)}\phi'(1)=\frac{a}{\ln(s_\delta/R)},$$

which implies

$$\frac{s_\delta}{\delta|\ln\delta|^{(p-1)/2}}\to\left(\frac{|\phi'(1)|}{a}\right)^{(p-1)/2}>0,\quad \text{as } \delta\to0.$$

Moreover, by Pohozaev identity, we can get that

$$\int_{B_1(0)}\phi^{p+1}=\frac{\pi(p+1)}{2}|\phi'(1)|^2 \text{ and } \int_{B_1(0)}\phi^p=2\pi|\phi'(1)|.$$

For any $z\in\Omega$, define $W_{\delta,z,a}(x)=W_{\delta,a}(x-z)$. Because $W_{\delta,z,a}$ does not vanish on $\partial\Omega$, we need to make a projection. Let $PW_{\delta,z,a}$ be the solution of

$$\begin{cases} -\delta^2 \Delta w = (W_{\delta,z,a} - a)_+^p, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases}$$

Then

$$PW_{\delta,z,a} = W_{\delta,z,a} - \frac{a}{\ln \frac{R}{s_\delta}} g(x,z), \tag{2.3}$$

where $g(x,z)$ satisfies

$$\begin{cases} -\Delta g = 0, & \text{in } \Omega, \\ g = \ln \frac{R}{|x-z|}, & \text{on } \partial\Omega. \end{cases}$$

It is easy to see that

$$g(x,z) = \ln R + 2\pi h(x,z),$$

where $h(x,z) = -H(x,z)$.

Let $Z = (Z_m^+, Z_n^-)$, where $Z_m^+ = (z_1^+, \cdots, z_m^+)$, $Z_n^+ = (z_1^-, \cdots, z_n^-)$. We will construct solutions for (1.11) of the form

$$\sum_{i=1}^m PW_{\delta, z_i^+, a_{\delta,i}^+} - \sum_{j=1}^n PW_{\delta, z_j^-, a_{\delta,j}^-} + \omega_\delta,$$

where $z_i^+, z_j^- \in \Omega$, $a_{\delta,i}^+ > 0, a_{\delta,j}^- > 0$ for $i = 1, \cdots, m$, $j = 1, \cdots, n$, ω_δ is a perturbation term. To make ω_δ as small as possible, we need to choose $a_{\delta,i}^+$, $a_{\delta,j}^-$ properly.

In this paper, we always assume that $z_i^+, z_j^- \in \Omega$ satisfies

$$\begin{aligned} & d(z_i^+, \partial\Omega) \geq \varrho,\ d(z_j^-, \partial\Omega) \geq \varrho, \quad |z_i^+ - z_k^+| \geq \varrho^{\bar{L}}, \quad i,k = 1, \cdots, m,\ i \neq k \\ & |z_j^- - z_l^-| \geq \varrho^{\bar{L}}, \quad |z_i^+ - z_j^-| \geq \varrho^{\bar{L}}, \quad j,l = 1, \cdots, n,\ j \neq l, \end{aligned} \tag{2.4}$$

where $\varrho > 0$ is a fixed small constant and $\bar{L} > 0$ is a fixed large constant.

Lemma 2.1. *For $\delta > 0$ small, there exist $(s_{\delta,1}^+(Z), \cdots, s_{\delta,m}^+(Z), s_{\delta,1}^-(Z), \cdots, s_{\delta,n}^-(Z))$ and $(a_{\delta,1}^+(Z), \cdots, a_{\delta,m}^+(Z), a_{\delta,1}^-(Z), \cdots, a_{\delta,n}^-(Z))$ satisfying the following system*

$$\delta^{2/(p-1)} (s_i^+)^{-2/(p-1)} \phi'(1) = \frac{a_i^+}{\ln(s_i^+/R)}, \qquad i = 1, \cdots, m \tag{2.5}$$

$$\delta^{2/(p-1)} (s_j^-)^{-2/(p-1)} \phi'(1) = \frac{a_j^-}{\ln(s_j^-/R)}, \qquad j = 1, \cdots, n \tag{2.6}$$

and

$$a_i^+ = \kappa_i^+ + \frac{2\pi q(z_i^+)}{|\ln\varepsilon|} + \frac{g(z_i^+, z_i^+)}{\ln \frac{R}{s_i^+}} a_i^+ - \sum_{\alpha \neq i}^m \frac{\bar{G}(z_i^+, z_\alpha^+)}{\ln \frac{R}{s_\alpha^+}} a_\alpha^+ + \sum_{l=1}^n \frac{\bar{G}(z_i^+, z_l^-)}{\ln \frac{R}{s_l^-}} a_l^-, \quad i = 1, \cdots, m, \tag{2.7}$$

$$a_j^- = \kappa_j^- - \frac{2\pi q(z_j^-)}{|\ln\varepsilon|} + \frac{g(z_i^-, z_i^-)}{\ln\frac{R}{s_j^-}} a_j^- - \sum_{\beta\neq j}^{n} \frac{\bar{G}(z_\beta^-, z_j^-)}{\ln\frac{R}{s_\beta^+}} a_\beta^+ + \sum_{k=1}^{m} \frac{\bar{G}(z_j^-, z_k^+)}{\ln\frac{R}{s_k^-}} a_k^-, \quad j = 1, \cdots, n, \tag{2.8}$$

where $\bar{G}(x,y) = \ln\frac{R}{|x-y|} - g(x,y)$ *for* $x \neq y$*.*

Since the proof is exactly the same as in Lemma 2.1 in [13], we omit it here therefore.

To simplify our notations, for given $Z = (Z_m^+, Z_n^-)$, in this paper, we will use $a_{\delta,i}^\pm$, $s_{\delta,i}^\pm$ instead of $a_{\delta,i}^\pm(Z)$, $s_{\delta,i}^\pm(Z)$. From now on we will always choose $(a_{\delta,1}^+, \cdots, a_{\delta,m}^+, a_{\delta,1}^-, \cdots, a_{\delta,n}^-)$ and $(s_{\delta,1}^+, \cdots, s_{\delta,m}^+, s_{\delta,1}^-, \cdots, s_{\delta,n}^-)$ such that (2.5)–(2.8) hold. For $(a_{\delta,1}^+, \cdots, a_{\delta,m}^+, a_{\delta,1}^-, \cdots, a_{\delta,n}^-)$ and $(s_{\delta,1}^+, \cdots, s_{\delta,m}^+, s_{\delta,1}^-, \cdots, s_{\delta,n}^-)$ chosen in such a way let us define

$$P_{\delta,Z,i}^+ = PW_{\delta, z_i^+, a_{\delta,i}^+}, \ P_{\delta,Z,j}^- = PW_{\delta, z_j^-, a_{\delta,j}^-}. \tag{2.9}$$

Remark 2.2. As in [13], we have the following asymptotic expansions:

$$\frac{1}{\ln\frac{R}{s_{\delta,i}^+}} = \frac{1}{\ln\frac{R}{\varepsilon}} + 0\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|^2}\right), \ i = 1, \cdots, m, \tag{2.10}$$

$$a_{\delta,i}^+ = 1 + \frac{2\pi q(z_i^+)}{\kappa|\ln\varepsilon|} + \frac{g(z_i^+, z_i^+)}{\ln\frac{R}{\varepsilon}} - \sum_{\alpha\neq i}^{m} \frac{\bar{G}(z_i^+, z_\alpha^+)}{\ln\frac{R}{\varepsilon}} + \sum_{l=1}^{n} \frac{\bar{G}(z_i^+, z_l^-)}{\ln\frac{R}{\varepsilon}} + 0\Big(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|^2}\Big), \ i = 1, \cdots, m, \tag{2.11}$$

$$\frac{\partial a_{\delta,i}^+}{\partial z_{k,h}^\pm} = 0\left(\frac{1}{|\ln\varepsilon|}\right), \ \frac{\partial s_{\delta,i}^+}{\partial z_{k,h}^\pm} = 0\left(\frac{\varepsilon}{|\ln\varepsilon|}\right), \ i = 1, \cdots, m, \ h = 1, 2. \tag{2.12}$$

Moreover, $a_{\delta,j}^-$ and $s_{\delta,j}^-$ have similar expansions.

To simplify notations, set

$$P_{\delta,Z}^+ = \sum_{\alpha=1}^{m} P_{\delta,Z,\alpha}^+, \ P_{\delta,Z}^- = \sum_{\beta=1}^{n} P_{\delta,Z,\beta}^-.$$

Then, we find that for $x \in B_{Ls_{\delta,i}^+}(z_i^+)$, where $L > 0$ is any fixed constant,

$$
\begin{aligned}
&P^+_{\delta,Z,i}(x)-\kappa^+_i-\frac{2\pi q(x)}{|\ln\varepsilon|}=W_{\delta,z^+_i,a^+_{\delta,i}}(x)-\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}g(x,z^+_i)-\kappa^+_i-\frac{2\pi q(x)}{|\ln\varepsilon|}\\
=&W_{\delta,z^+_i,a^+_{\delta,i}}(x)-\kappa^+_i-\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}g(z^+_i,z^+_i)-\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}\Big(\big\langle Dg(z^+_i,z^+_i),x-z^+_i\big\rangle+O(|x-z^+_i|^2)\Big)\\
&-\frac{2\pi q(z^+_i)}{|\ln\varepsilon|}-\frac{2\pi}{|\ln\varepsilon|}\left(\big\langle Dq(z^+_i),x-z^+_i\big\rangle+O(|x-z^+_i|^2)\right)\\
=&W_{\delta,z^+_i,a^+_{\delta,i}}(x)-\kappa^+_i-\frac{2\pi q(z^+_i)}{|\ln\varepsilon|}-\frac{2\pi}{|\ln\varepsilon|}\big\langle Dq(z^+_i),x-z^+_i\big\rangle\\
&-\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}g(z^+_i,z^+_i)-\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}\big\langle Dg(z^+_i,z^+_i),x-z^+_i\big\rangle+O\left(\frac{(s^+_{\delta,i})^2}{|\ln\varepsilon|}\right),
\end{aligned}
$$

and for $k\neq i$ and $x\in B_{Ls^+_{\delta,i}}(z^+_i)$, by (2.2)

$$
\begin{aligned}
P^+_{\delta,Z,k}(x)&=W_{\delta,z^+_k,a^+_{\delta,k}}(x)-\frac{a^+_{\delta,k}}{\ln\frac{R}{s^+_{\delta,k}}}g(x,z^+_k)=\frac{a^+_{\delta,k}}{\ln\frac{R}{s^+_{\delta,k}}}\bar G(x,z^+_k)\\
&=\frac{a^+_{\delta,k}}{\ln\frac{R}{s^+_{\delta,k}}}\bar G(z^+_i,z^+_k)+\frac{a^+_{\delta,k}}{\ln\frac{R}{s^+_{\delta,k}}}\big\langle D\bar G(z^+_i,z^+_k),x-z^+_i\big\rangle+O\Big(\frac{(s^+_{\delta,i})^2}{|\ln\varepsilon|}\Big)
\end{aligned}
$$

and

$$
P^-_{\delta,Z,j}(x)=\frac{a^-_{\delta,j}}{\ln\frac{R}{s^-_{\delta,j}}}\bar G(z^+_i,z^-_j)+\frac{a^-_{\delta,j}}{\ln\frac{R}{s^-_{\delta,j}}}\big\langle D\bar G(z^+_i,z^-_j),x-z^+_i\big\rangle+O\Big(\frac{(s^+_{\delta,i})^2}{|\ln\varepsilon|}\Big).
$$

So, by using (2.7), we obtain

$$
\begin{aligned}
&P^+_{\delta,Z}(x)-P^-_{\delta,Z}(x)-\kappa^+_i-\frac{2\pi q(x)}{|\ln\varepsilon|}\\
=&W_{\delta,z^+_i,a^+_{\delta,i}}(x)-a^+_{\delta,i}-\frac{2\pi}{|\ln\varepsilon|}\big\langle Dq(z^+_i),x-z^+_i\big\rangle-\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}\big\langle Dg(z^+_i,z^+_i),x-z^+_i\big\rangle\\
&+\sum_{k\neq i}^{m}\frac{a^+_{\delta,k}}{\ln\frac{R}{s^+_{\delta,k}}}\big\langle D\bar G(z^+_i,z^+_k),x-z^+_i\big\rangle-\sum_{l=1}^{n}\frac{a^-_{\delta,l}}{\ln\frac{R}{s^-_{\delta,l}}}\big\langle D\bar G(z^+_i,z^-_l),x-z^+_i\big\rangle\\
&+O\left(\frac{(s^+_{\delta,i})^2}{|\ln\varepsilon|}\right),\quad x\in B_{Ls^+_{\delta,i}}(z^+_i).
\end{aligned}
\tag{2.13}
$$

Similarly, we have

$$\begin{aligned}
&P^-_{\delta,Z}(x)-P^+_{\delta,Z}(x)-\kappa^-_j+\frac{2\pi q(x)}{|\ln\varepsilon|}\\
=&W_{\delta,z^-_j,a^-_{\delta,j}}(x)-a^-_{\delta,j}+\frac{2\pi}{|\ln\varepsilon|}\left\langle Dq(z^-_j),x-z^-_j\right\rangle-\frac{a^-_{\delta,j}}{\ln\frac{R}{s^-_{\delta,j}}}\left\langle Dg(z^-_j,z^-_j),x-z^-_j\right\rangle\\
&+\sum_{l\neq j}^{n}\frac{a^-_{\delta,l}}{\ln\frac{R}{s^-_{\delta,l}}}\left\langle D\bar G(z^-_j,z^-_l),x-z^-_j\right\rangle-\sum_{k=1}^{m}\frac{a^+_{\delta,k}}{\ln\frac{R}{s^+_{\delta,k}}}\left\langle D\bar G(z^-_j,z^+_k),x-z^-_j\right\rangle\\
&+O\left(\frac{(s^-_{\delta,j})^2}{|\ln\varepsilon|}\right),\quad x\in B_{Ls^-_{\delta,j}}(z^-_j).
\end{aligned}\tag{2.14}$$

We end this section by giving the following formula which can be obtained by direct computation and will be used in the next two sections.

$$\begin{aligned}
&\frac{\partial W_{\delta,z^\pm_i,a^\pm_{\delta,i}}(x)}{\partial z^\pm_{i,h}}\\
=&\begin{cases}\dfrac{1}{\delta}\Big(\dfrac{a^\pm_{\delta,i}}{|\phi'(1)||\ln\frac{R}{s^\pm_{\delta,i}}|}\Big)^{(p+1)/2}\phi'\big(\dfrac{|x-z^\pm_i|}{s^\pm_{\delta,i}}\big)\dfrac{z^\pm_{i,h}-x_h}{|x-z^\pm_i|}+O\left(\dfrac{1}{|\ln\varepsilon|}\right), & x\in B_{s^\pm_{\delta,i}}(z^\pm_i),\\[2ex]
-\dfrac{a^\pm_{\delta,i}}{\ln\frac{R}{s^\pm_{\delta,i}}}\dfrac{z^\pm_{i,h}-x_h}{|x-z^\pm_i|^2}+O\left(\dfrac{1}{|\ln\varepsilon|}\right), & x\in\Omega\setminus B_{s^\pm_{\delta,i}}(z^\pm_i).\end{cases}
\end{aligned}\tag{2.15}$$

3. THE REDUCTION

Let

$$w(x)=\begin{cases}\phi(|x|), & |x|\le 1,\\ \phi'(1)\ln|x|, & |x|>1.\end{cases}$$

Then $w\in C^1(\mathbb{R}^2)$. Since $\phi'(1)<0$ and $\ln|x|$ is harmonic for $|x|>1$, we see that w satisfies

$$-\Delta w=w^p_+,\quad \text{in }\mathbb{R}^2.\tag{3.1}$$

Moreover, since w_+ is Lip-continuous, by the Schauder estimate, $w\in C^{2,\alpha}$ for any $\alpha\in(0,1)$.

Consider the following problem:

$$-\Delta v-pw^{p-1}_+v=0,\quad v\in L^\infty(\mathbb{R}^2),\tag{3.2}$$

It is easy to see that $\frac{\partial w}{\partial x_i}$, $i=1,2$, is a solution of (3.2). Moreover, from Dancer and Yan [15], we know that w is also non-degenerate, in the sense that the kernel of the operator $Lv := -\Delta v - pw_+^{p-1}v,\ v\in D^{1,2}(\mathbb{R}^2)$ is spanned by $\{\frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}\}$.

Let $P^+_{\delta,Z,i},\ P^-_{\delta,Z,j}$ be the functions defined in (2.9). Set

$$F_{\delta,Z}=\Big\{u: u\in L^p(\Omega), \int_\Omega \frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}}u=0, \int_\Omega \frac{\partial P^-_{\delta,Z,j}}{\partial z^-_{j,h}}u=0,$$
$$i=1,\cdots,m,\quad j=1,\cdots,n,\quad h=1,2\Big\},$$

and

$$E_{\delta,Z}=\Big\{u:\ u\in W^{2,p}(\Omega)\cap H^1_0(\Omega), \int_\Omega \Delta\left(\frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}}\right)u=0,\ \int_\Omega \Delta\left(\frac{\partial P^-_{\delta,Z,j}}{\partial z^-_{j,h}}\right)u=0,$$
$$i=1,\cdots,m,\quad j=1,\cdots,n,\quad h=1,2\Big\}.$$

For any $u\in L^p(\Omega)$, define $Q_\delta u$ as follows:

$$Q_\delta u = u-\sum_{i=1}^m\sum_{h=1}^2 b^+_{i,h}\left(-\delta^2\Delta\Big(\frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}}\Big)\right)-\sum_{j=1}^n\sum_{\bar h=1}^2 b^-_{j,\bar h}\left(-\delta^2\Delta\Big(\frac{\partial P^-_{\delta,Z,j}}{\partial z^-_{j,\bar h}}\Big)\right),$$

where the constants $b^+_{i,h}$, $b^-_{j,\bar h}$ satisfy

$$\begin{aligned}&\sum_{i=1}^m\sum_{h=1}^2 b^+_{i,h}\left(-\delta^2\int_\Omega\Delta\Big(\frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}}\Big)\frac{\partial P^+_{\delta,Z,k}}{\partial z^+_{k,\hat h}}\right)\\ &+\sum_{j=1}^n\sum_{\bar h=1}^2 b^-_{j,\bar h}\left(-\delta^2\int_\Omega\Delta\Big(\frac{\partial P^-_{\delta,Z,j}}{\partial z^-_{j,\bar h}}\Big)\frac{\partial P^+_{\delta,Z,k}}{\partial z^+_{k,\hat h}}\right)=\int_\Omega u\frac{\partial P^+_{\delta,Z,k}}{\partial z^+_{k,\hat h}},\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}&\sum_{i=1}^m\sum_{h=1}^2 b^+_{i,h}\left(-\delta^2\int_\Omega\Delta\Big(\frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}}\Big)\frac{\partial P^-_{\delta,Z,l}}{\partial z^-_{l,\tilde h}}\right)\\ &+\sum_{j=1}^n\sum_{\bar h=1}^2 b^-_{j,\bar h}\left(-\delta^2\int_\Omega\Delta\Big(\frac{\partial P^-_{\delta,Z,j}}{\partial z^-_{j,\bar h}}\Big)\frac{\partial P^-_{\delta,Z,l}}{\partial z^-_{l,\tilde h}}\right)=\int_\Omega u\frac{\partial P^-_{\delta,Z,l}}{\partial z^-_{l,\tilde h}}.\end{aligned}\tag{3.4}$$

Since $\int_\Omega \frac{\partial P^+_{\delta,Z,k}}{\partial z^+_{k,\hat h}} Q_\delta u = 0$, $\int_\Omega \frac{\partial P^-_{\delta,Z,l}}{\partial z^-_{l,\tilde h}} Q_\delta u = 0$, the operator Q_δ can be regarded as a projection from $L^p(\Omega)$ to $F_{\delta,Z}$. In order to show that we can solve (3.3) and (3.4) to obtain $b^+_{i,h}$ and $b^-_{j,\bar h}$, we just need the following estimate (by (2.12) and (2.15)):

$$\begin{aligned}
& -\delta^2 \int_\Omega \Delta\Big(\frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}}\Big) \frac{\partial P^+_{\delta,Z,k}}{\partial z^+_{k,\hat h}} \\
=& p\int_\Omega (W_{\delta,z_i^+,a^+_{\delta,i}} - a^+_{\delta,i})^{p-1}_+ \left(\frac{\partial W_{\delta,z_i^+,a^+_{\delta,i}}}{\partial z^+_{i,h}} - \frac{\partial a^+_{\delta,i}}{\partial z^+_{i,h}}\right) \frac{\partial P^+_{\delta,Z,k}}{\partial z^+_{k,\hat h}} \\
=& \delta_{ikh\hat h} \frac{c}{|\ln\varepsilon|^{p+1}} + 0\left(\frac{\varepsilon}{|\ln\varepsilon|^{p+1}}\right),
\end{aligned} \tag{3.5}$$

where $c > 0$ is a constant, $\delta_{ikh\hat h} = 1$, if $i = k$ and $h = \hat h$; otherwise, $\delta_{ijh\hat h} = 0$.

Similarly,

$$-\delta^2 \int_\Omega \Delta\Big(\frac{\partial P^-_{\delta,Z,j}}{\partial z^-_{j,\bar h}}\Big) \frac{\partial P^-_{\delta,Z,l}}{\partial z^-_{l,\tilde h}} = \delta_{jl\bar h\tilde h} \frac{c}{|\ln\varepsilon|^{p+1}} + 0\left(\frac{\varepsilon}{|\ln\varepsilon|^{p+1}}\right), \tag{3.6}$$

where $c > 0$ is a constant, $\delta_{jl\bar h\tilde h} = 1$, if $j = l$ and $\bar h = \tilde h$; otherwise, $\delta_{jl\bar h\tilde h} = 0$.

Set

$$\begin{aligned}
L_\delta u = -\delta^2 \Delta u - \sum_{i=1}^m p\chi_{\Omega_i^+}\left(P^+_{\delta,Z} - P^-_{\delta,Z} - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)^{p-1}_+ u \\
- \sum_{j=1}^m p\chi_{\Omega_j^-}\left(P^-_{\delta,Z} - P^+_{\delta,Z} - \kappa_j^- + \frac{2\pi q(x)}{|\ln\varepsilon|}\right)^{p-1}_+ u,
\end{aligned}$$

and

$$B_{\delta,Z} = \Big(\cup_{i=1}^m B_{Ls^+_{\delta,i}}(z_i^+)\Big) \bigcup \Big(\cup_{j=1}^n B_{Ls^-_{\delta,j}}(z_j^-)\Big).$$

We have the following lemma.

Lemma 3.1. *There are constants $\rho_0 > 0$ and $\delta_0 > 0$, such that for any $\delta \in (0, \delta_0]$, Z satisfying (2.4), $u \in E_{\delta,Z}$ with $Q_\delta L_\delta u = 0$ in $\Omega \setminus B_{\delta,Z}$ for some $L > 0$ large, then*

$$\|Q_\delta L_\delta u\|_{L^p(\Omega)} \ge \frac{\rho_0 \delta^{\frac{2}{p}}}{|\ln\delta|^{\frac{(p-1)^2}{p}}} \|u\|_{L^\infty(\Omega)}.$$

Proof. Set $s^\pm_{N,j} = s^\pm_{\delta_N,j}$. In the sequel, we will use $\|\cdot\|_p, \|\cdot\|_\infty$ to denote $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{L^\infty(\Omega)}$ respectively.

We argue by contradiction. Suppose that there are $\delta_N \to 0$, Z_N satisfying (2.4) and $u_N \in E_{\delta_N, Z_N}$ with $Q_{\delta_N} L_{\delta_N} u_N = 0$ in $\Omega \setminus B_{\delta_N, Z_N}$ and $\|u_N\|_\infty = 1$ such that

$$\|Q_{\delta_N} L_{\delta_N} u_N\|_p \leq \frac{1}{N} \frac{\delta_N^{\frac{2}{p}}}{|\ln \delta_N|^{\frac{(p-1)^2}{p}}}.$$

First, we estimate $b^+_{i,h,N}$ and $b^-_{j,\bar{h},N}$ in the following formula:

$$\begin{aligned} Q_{\delta_N} L_{\delta_N} u_N = L_{\delta_N} u_N &- \sum_{i=1}^{m} \sum_{h=1}^{2} b^+_{i,h,N} \left(-\delta_N^2 \Delta \frac{\partial P^+_{\delta_N, Z_N, i}}{\partial z^+_{i,h}} \right) \\ &- \sum_{j=1}^{n} \sum_{\bar{h}=1}^{2} b^-_{j,\bar{h},N} \left(-\delta_N^2 \Delta \frac{\partial P^-_{\delta_N, Z_N, j}}{\partial z^-_{j,\bar{h}}} \right). \end{aligned} \tag{3.7}$$

For each fixed k, multiplying (3.7) by $\frac{\partial P^+_{\delta_N, Z_N, k}}{\partial z^+_{k,\hat{h}}}$, noting that

$$\int_\Omega \left(Q_{\delta_N} L_{\delta_N} u_N \right) \frac{\partial P^+_{\delta_N, Z_N, k}}{\partial z^+_{k,\hat{h}}} = 0,$$

we obtain

$$\begin{aligned} &\int_\Omega u_N \, L_{\delta_N} \left(\frac{\partial P^+_{\delta_N, Z_N, k}}{\partial z^+_{k,\hat{h}}} \right) = \int_\Omega \left(L_{\delta_N} u_N \right) \frac{\partial P^+_{\delta_N, Z_N, k}}{\partial z^+_{k,\hat{h}}} \\ &= \sum_{i=1}^{m} \sum_{h=1}^{2} b^+_{i,h,N} \int_\Omega \left(-\delta_N^2 \Delta \frac{\partial P^+_{\delta_N, Z_N, i}}{\partial z^+_{i,h}} \right) \frac{\partial P^+_{\delta_N, Z_N, k}}{\partial z^+_{k,\hat{h}}} \\ &\quad + \sum_{j=1}^{n} \sum_{\bar{h}=1}^{2} b^-_{j,\bar{h},N} \int_\Omega \left(-\delta_N^2 \Delta \frac{\partial P^-_{\delta_N, Z_N, j}}{\partial z^-_{j,\bar{h}}} \right) \frac{\partial P^+_{\delta_N, Z_N, k}}{\partial z^+_{k,\hat{h}}}. \end{aligned}$$

Using (2.13), (2.14) and Lemma A.1, we obtain

$$
\begin{aligned}
&\int_\Omega u_N\, L_{\delta_N}\left(\frac{\partial P^+_{\delta_N,Z_N,k}}{\partial z^+_{k,\hat h}}\right)\\
=&\int_\Omega\Bigg[-\delta_N^2\Delta\left(\frac{\partial P^+_{\delta_N,Z_N,k}}{\partial z^+_{k,\hat h}}\right)-\sum_{i=1}^m p\chi_{\Omega_i^+}\left(P^+_{\delta_N,Z_N}-P^-_{\delta_N,Z_N}-\kappa_i^+-\frac{2\pi q(x)}{|\ln\varepsilon_N|}\right)_+^{p-1}\frac{\partial P^+_{\delta_N,Z_N,k}}{\partial z^+_{k,\hat h}}\\
&-\sum_{j=1}^n p\chi_{\Omega_j^-}\left(P^-_{\delta_N,Z_N}-P^+_{\delta_N,Z_N}-1+\frac{2\pi q(x)}{\kappa|\ln\varepsilon_N|}\right)_+^{p-1}\frac{\partial P^+_{\delta_N,Z_N,k}}{\partial z^+_{k,\hat h}}\Bigg]u_N\\
=&\,p\int_\Omega\left(W_{\delta_N,z^+_{k,N},a^+_{\delta_N,k}}-a^+_{\delta_N,k}\right)_+^{p-1}\left(\frac{\partial W_{\delta_N,z^+_{k,N},a^+_{\delta_N,k}}}{\partial z^+_{k,\hat h}}-\frac{\partial a^+_{\delta_N,k}}{\partial z^+_{k,\hat h}}\right)u_N\\
&-p\sum_{\alpha=1}^m\int_{\Omega_\alpha^+}\left(W_{\delta_N,z^+_{\alpha,N},a^+_{\delta_N,\alpha}}-a^+_{\delta_N,\alpha}+O\left(\frac{s^+_{N,\alpha}}{|\ln\varepsilon_N|}\right)\right)_+^{p-1}\frac{\partial P^+_{\delta_N,Z_N,k}}{\partial z^+_{k,\hat h}}u_N\\
&-p\sum_{\beta=1}^n\int_{\Omega_\beta^-}\left(W_{\delta_N,z^-_{\beta,N},a^-_{\delta_N,\beta}}-a^-_{\delta_N,\beta}+O\left(\frac{s^-_{N,\beta}}{|\ln\varepsilon_N|}\right)\right)_+^{p-1}\frac{\partial P^+_{\delta_N,Z_N,k}}{\partial z^+_{k,\hat h}}u_N\\
=&\,0\left(\frac{\varepsilon_N^2}{|\ln\varepsilon_N|^p}\right).
\end{aligned}
$$

Using (3.5) and (3.6), we find that

$$
b^+_{i,h,N}=0\left(\varepsilon_N^2|\ln\varepsilon_N|\right).
$$

Similarly,

$$
b^-_{i,h,N}=0\left(\varepsilon_N^2|\ln\varepsilon_N|\right).
$$

Therefore,

$$\begin{aligned}
&\sum_{i=1}^{m}\sum_{h=1}^{2} b_{i,h,N}^{+}\left(-\delta_N^2\Delta\frac{\partial P_{\delta_N,Z_N,i}^{+}}{\partial z_{i,h}^{+}}\right)+\sum_{j=1}^{n}\sum_{\bar h=1}^{2} b_{j,\bar h,N}^{-}\left(-\delta_N^2\Delta\frac{\partial P_{\delta_N,Z_N,j}^{-}}{\partial z_{j,\bar h}^{-}}\right)\\
&=p\sum_{i=1}^{m}\sum_{h=1}^{2} b_{i,h,N}^{+}\left(W_{\delta_N,z_{i,N}^{+},a_{\delta_N,i}^{+}}-a_{\delta_N,i}^{+}\right)_+^{p-1}\left(\frac{\partial W_{\delta_N,z_{i,N}^{+},a_{\delta_N,i}^{+}}}{\partial z_{i,h}^{+}}-\frac{\partial a_{\delta_N,i}^{+}}{\partial z_{i,h}^{+}}\right)\\
&\quad+p\sum_{j=1}^{n}\sum_{\bar h=1}^{2} b_{j,\bar h,N}^{-}\left(W_{\delta_N,z_{j,N}^{-},a_{\delta_N,j}^{-}}-a_{\delta_N,j}^{-}\right)_+^{p-1}\left(\frac{\partial W_{\delta_N,z_{j,N}^{-},a_{\delta_N,j}^{-}}}{\partial z_{j,\bar h}^{-}}-\frac{\partial a_{\delta_N,j}^{-}}{\partial z_{j,\bar h}^{-}}\right)\\
&=0\left(\sum_{i=1}^{m}\sum_{h=1}^{2}\frac{\varepsilon_N^{\frac{2}{p}-1}|b_{i,h,N}^{+}|}{|\ln\varepsilon_N|^{p}}\right)+0\left(\sum_{j=1}^{n}\sum_{\bar h=1}^{2}\frac{\varepsilon_N^{\frac{2}{p}-1}|b_{j,\bar h,N}^{-}|}{|\ln\varepsilon_N|^{p}}\right)\\
&=0\left(\frac{\varepsilon_N^{\frac{2}{p}+1}}{|\ln\varepsilon_N|^{p-1}}\right)\quad \text{in } L^p(\Omega).
\end{aligned}$$

Thus, we obtain

$$L_{\delta_N}u_N=Q_{\delta_N}L_{\delta_N}u_N+O\left(\frac{\varepsilon_N^{\frac{2}{p}+1}}{|\ln\varepsilon_N|^{p-1}}\right)=O\left(\frac{1}{N}\frac{\delta_N^{\frac{2}{p}}}{|\ln\delta_N|^{\frac{(p-1)^2}{p}}}\right).$$

For any fixed i,j, define

$$\tilde u_{i,N}^{+}(y)=u_N(s_{N,i}^{+}y+z_{i,N}^{+}),\quad \tilde u_{j,N}^{-}(y)=u_N(s_{N,j}^{-}y+z_{j,N}^{-}).$$

Let

$$\begin{aligned}
\tilde L_N^{\pm}u=&-\Delta u-\sum_{k=1}^{m}p\frac{(s_{N,i}^{\pm})^2}{\delta_N^2}\chi_{\Omega_k^{+}}\left(P_{\delta_N,Z_N}^{+}(s_{N,i}^{\pm}y+z_{i,N}^{\pm})-P_{\delta_N,Z_N}^{-}(s_{N,i}^{\pm}y+z_{i,N}^{\pm})-\kappa_k^{+}-\frac{2\pi q}{|\ln\varepsilon_N|}\right)_+^{p-1}u\\
&-\sum_{l=1}^{n}p\frac{(s_{N,i}^{\pm})^2}{\delta_N^2}\chi_{\Omega_l^{-}}\left(P_{\delta_N,Z_N}^{-}(s_{N,i}^{\pm}y+z_{i,N}^{\pm})-P_{\delta_N,Z_N}^{+}(s_{N,i}^{\pm}y+z_{i,N}^{\pm})-\kappa_l^{-}+\frac{2\pi q}{|\ln\varepsilon_N|}\right)_+^{p-1}u.
\end{aligned}$$

Then

$$(s_{N,i}^{\pm})^{\frac{2}{p}}\times\frac{\delta_N^2}{(s_{N,i}^{\pm})^2}\|\tilde L_N^{\pm}\tilde u_{i,N}^{\pm}\|_p=\|L_{\delta_N}u_N\|_p.$$

Noting that

$$\left(\frac{\delta_N}{s_{N,i}^{\pm}}\right)^2=O\left(\frac{1}{|\ln\delta_N|^{p-1}}\right),$$

we find that

$$L_{\delta_N} u_N = o\left(\frac{\delta_N^{\frac{2}{p}}}{|\ln \delta_N|^{\frac{(p-1)^2}{p}}}\right).$$

As a result,

$$\tilde{L}_N^{\pm} \tilde{u}_{i,N}^{\pm} = o(1), \quad \text{in } L^p(\Omega_N^{\pm}),$$

where $\Omega_N^{\pm} = \left\{y : s_{N,i}^{\pm} y + z_{i,N}^{\pm} \in \Omega\right\}$.

Since $\|\tilde{u}_{i,N}^{\pm}\|_\infty = 1$, by the regularity theory of elliptic equations, we may assume that

$$\tilde{u}_{i,N}^{\pm} \to u_i^{\pm}, \quad \text{in } C_{loc}^1(\mathbb{R}^2).$$

It is easy to see that

$$\begin{aligned}
&\sum_{k=1}^{m} \frac{(s_{N,i}^{+})^2}{\delta_N^2} \chi_{\Omega_k^+} \left(P_{\delta_N,Z_N}^{+}(s_{N,i}^{+} y + z_{i,N}^{+}) - P_{\delta_N,Z_N}^{-}(s_{N,i}^{+} y + z_{i,N}^{+}) - \kappa_k^{+} - \frac{2\pi q}{|\ln \varepsilon_N|}\right)_+^{p-1} \\
&= \frac{(s_{N,i}^{+})^2}{\delta_N^2} \left(W_{\delta_N, z_{i,N}^{+}, a_{\delta_N,i}^{+}} - a_{\delta_N,i}^{+} + O\left(\frac{s_{N,i}^{+}}{|\ln \varepsilon_N|}\right)\right)_+^{p-1} + o(1) \\
&\to w_+^{p-1}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{l=1}^{n} \frac{(s_{N,j}^{-})^2}{\delta_N^2} \chi_{\Omega_l^-} \left(P_{\delta_N,Z_N}^{-}(s_{N,j}^{-} y + z_{j,N}^{-}) - P_{\delta_N,Z_N}^{+}(s_{N,j}^{-} y + z_{j,N}^{-}) - \kappa_l^{-} + \frac{2\pi q}{|\ln \varepsilon_N|}\right)_+^{p-1} \\
&\to w_+^{p-1}.
\end{aligned}$$

Then, by Lemma A.1, we find that $u_i^{\pm}$ satisfies

$$-\Delta u - p w_+^{p-1} u = 0.$$

Now from the Proposition 3.1 in [15], we have

$$u_i^{\pm} = c_1^{\pm} \frac{\partial w}{\partial x_1} + c_2^{\pm} \frac{\partial w}{\partial x_2}. \tag{3.8}$$

Since

$$\int_\Omega \Delta\Big(\frac{\partial P_{\delta_N,Z_N,i}^{\pm}}{\partial z_{i,h}^{\pm}}\Big) u_N = 0,$$

we find that

$$\int_{\mathbb{R}^2} \phi_+^{p-1} \frac{\partial \phi}{\partial z_h} u_i^{\pm} = 0,$$

which, together with (3.8), gives $u_i^{\pm} \neq 0$. Thus,

$$\tilde{u}^{\pm}_{i,N} \to 0, \quad \text{in } C^1(B_L(0)),$$

for any $L > 0$, which implies that $u_N = o(1)$ on $\partial B_{Ls^{\pm}_{N,i}}(z^{\pm}_{i,N})$.

By assumption,

$$Q_{\delta_N} L_{\delta_N} u_N = 0, \quad \text{in } \Omega \setminus B_{\delta_N, Z_N}.$$

On the other hand, by Lemma A.1, for $i = 1, \cdots, m$, $j = 1, \cdots, n$, we have

$$\left(P^{+}_{\delta_N,Z_N} - P^{-}_{\delta_N,Z_N} - \kappa^{+}_i - \frac{2\pi q(x)}{|\ln \varepsilon_N|}\right)_+ = 0, \quad x \in \Omega^{+}_i \setminus B_{Ls^{+}_{N,i}}(z^{+}_{i,N}),$$

$$\left(P^{-}_{\delta_N,Z_N} - P^{+}_{\delta_N,Z_N} - \kappa^{-}_j + \frac{2\pi q(x)}{|\ln \varepsilon_N|}\right)_+ = 0, \quad x \in \Omega^{-}_j \setminus B_{Ls^{-}_{N,j}}(z^{-}_{j,N}).$$

Thus, we find that

$$-\Delta u_N = 0, \quad \text{in } \Omega \setminus B_{\delta_N, Z_N}.$$

However, $u_N = 0$ on $\partial\Omega$ and $u_N = o(1)$ on $\partial B_{\delta_N, Z_N}$. So we have

$$u_N = o(1).$$

This is a contradiction.

□

From Lemma 3.1, using Fredholm alternative, we can prove, as in [13], the following result:

Proposition 3.2. *$Q_\delta L_\delta$ is one to one and onto from $E_{\delta,Z}$ to $F_{\delta,Z}$.*

Now consider the equation

$$Q_\delta L_\delta \omega = Q_\delta l^{+}_\delta - Q_\delta l^{-}_\delta + Q_\delta R^{+}_\delta(\omega) - Q_\delta R^{-}_\delta(\omega), \tag{3.9}$$

where

$$l^{+}_\delta = \sum_{i=1}^{m} \chi_{\Omega^{+}_i} \left(P^{+}_{\delta,Z} - P^{-}_{\delta,Z} - \kappa^{+}_i - \frac{2\pi q(x)}{|\ln \varepsilon|}\right)^p_+ - \sum_{i=1}^{m} \left(W_{\delta, z^{+}_i, a^{+}_{\delta,i}} - a^{+}_{\delta,i}\right)^p_+, \tag{3.10}$$

$$l^{-}_\delta = \sum_{j=1}^{n} \chi_{\Omega^{-}_j} \left(P^{-}_{\delta,Z} - P^{+}_{\delta,Z} - \kappa^{-}_j + \frac{2\pi q(x)}{|\ln \varepsilon|}\right)^p_+ - \sum_{j=1}^{n} \left(W_{\delta, z^{-}_j, a^{-}_{\delta,j}} - a^{-}_{\delta,j}\right)^p_+, \tag{3.11}$$

and

$$
\begin{aligned}
R_\delta^+(\omega) = &\sum_{i=1}^m \chi_{\Omega_i^+}\Bigg[\left(P_{\delta,Z}^+ - P_{\delta,Z}^- + \omega - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p - \left(P_{\delta,Z}^+ - P_{\delta,Z}^- - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p \\
&- p\left(P_{\delta,Z}^+ - P_{\delta,Z}^- - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p-1}\omega\Bigg],
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
R_\delta^-(\omega) = &\sum_{j=1}^n \chi_{\Omega_j^-}\Bigg[\left(P_{\delta,Z}^- - P_{\delta,Z}^+ - \omega - \kappa_j^- + \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p - \left(P_{\delta,Z}^- - P_{\delta,Z}^+ - \kappa_j^- + \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p \\
&+ p\left(P_{\delta,Z}^- - P_{\delta,Z}^+ - \kappa_j^- + \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p-1}\omega\Bigg].
\end{aligned}
\tag{3.13}
$$

Using Proposition 3.2, we can rewrite (3.9) as

$$
\omega = G_\delta\omega =: (Q_\delta L_\delta)^{-1}Q_\delta\big(l_\delta^+ - l_\delta^- + R_\delta^+(\omega) - R_\delta^-(\omega)\big). \tag{3.14}
$$

The next Proposition enables us to reduce the problem of finding a solution for (1.11) to a finite dimensional problem.

Proposition 3.3. *There is an $\delta_0 > 0$, such that for any $\delta \in (0, \delta_0]$ and Z satisfying* (2.4), (3.9) *has a unique solution $\omega_\delta \in E_{\delta,Z}$, with*

$$
\|\omega_\delta\|_\infty = 0\Big(\delta|\ln\delta|^{\frac{p-1}{2}}\Big).
$$

Proof. It follows from Lemma A.1 that if L is large enough, δ is small then

$$
\left(P_{\delta,Z}^+ - P_{\delta,Z}^- - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+ = 0, \quad x \in \Omega_i^+ \setminus B_{Ls_{\delta,i}^+}(z_i^+),\ i = 1, \cdots, m
$$

$$
\left(P_{\delta,Z}^- - P_{\delta,Z}^+ - \kappa_j^- + \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+ = 0, \quad x \in \Omega_j^- \setminus B_{Ls_{\delta,j}^-}(z_j^-),\ j = 1, \cdots, n.
$$

Let

$$
M = E_{\delta,Z} \cap \Big\{\|\omega\|_\infty \le \delta|\ln\delta|^{\frac{p-1}{2}}\Big\}.
$$

Then M is complete under L^∞ norm and G_δ is a map from $E_{\delta,Z}$ to $E_{\delta,Z}$. We will show that G_δ is a contraction map from M to M.

Step 1. G_δ is a map from M to M.

For any $\omega \in M$, similar to Lemma A.1, it is easy to prove that for large $L > 0$, δ small

$$
\begin{aligned}
&\Big(P^+_{\delta,Z}-P^-_{\delta,Z}+\omega-\kappa^+_i-\frac{2\pi q(x)}{|\ln\varepsilon|}\Big)_+=0,\quad x\in\Omega^+_i\setminus B_{Ls^+_{\delta,i}}(z^+_i),\\
&\Big(P^-_{\delta,Z}-P^+_{\delta,Z}-\omega-\kappa^-_j+\frac{2\pi q(x)}{|\ln\varepsilon|}\Big)_+=0,\quad x\in\Omega^-_j\setminus B_{Ls^-_{\delta,j}}(z^-_j).
\end{aligned}
\tag{3.15}
$$

Note also that for any $u\in L^\infty(\Omega)$,

$$
Q_\delta u=u\quad\text{in }\Omega\setminus B_{\delta,Z}.
$$

Therefore, using Lemma A.1, (3.10)–(3.13), we find that for any $\omega\in M$,

$$
\begin{aligned}
&Q_\delta(l^+_\delta-l^-_\delta)+Q_\delta(R^+_\delta(\omega)-R^-_\delta(\omega))\\
=&l^+_\delta-l^-_\delta+R^+_\delta(\omega)-R^-_\delta(\omega)\\
=&0,\quad\text{in }\Omega\setminus B_{Z,\delta}.
\end{aligned}
$$

So, we can apply Lemma 3.1 to obtain

$$
\begin{aligned}
&\|(Q_\delta L_\delta)^{-1}\big(Q_\delta(l^+_\delta-l^-_\delta)+Q_\delta(R^+_\delta(\omega)-R^-_\delta(\omega))\big)\|_\infty\\
\le&\frac{C|\ln\delta|^{\frac{(p-1)^2}{p}}}{\delta^{\frac{2}{p}}}\|Q_\delta(l^+_\delta-l^-_\delta)+Q_\delta(R^+_\delta(\omega)-R^-_\delta(\omega))\|_p.
\end{aligned}
$$

Thus, for any $\omega\in M$, we have

$$
\begin{aligned}
\|G_\delta(\omega)\|_\infty&=\|(Q_\delta L_\delta)^{-1}Q_\delta\big(l^+_\delta-l^-_\delta+R^+_\delta(\omega)-R^-_\delta(\omega)\big)\|_\infty\\
&\le\tfrac{C|\ln\delta|^{\frac{(p-1)^2}{p}}}{\delta^{\frac{2}{p}}}\|Q_\delta\big(l^+_\delta-l^-_\delta+R^+_\delta(\omega)-R^-_\delta(\omega)\big)\|_p.
\end{aligned}
\tag{3.16}
$$

It follows from (3.3)–(3.6) that the constant $b^\pm_{k,\hat h}$, corresponding to $u\in L^\infty(\Omega)$, satisfies

$$
|b^\pm_{k,\hat h}|\le C|\ln\delta|^{p+1}\Bigg(\sum_{i,h}\int_\Omega\Big|\frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}}\Big||u|+\sum_{j,\bar h}\int_\Omega\Big|\frac{\partial P^-_{\delta,Z,j}}{\partial z^-_{j,\bar h}}\Big||u|\Bigg).
$$

Since

$$
l^+_\delta-l^-_\delta+R^+_\delta(\omega)-R^-_\delta(\omega)=0,\quad\text{in }\Omega\setminus B_{\delta,Z},
$$

we find that the constant $b^\pm_{k,\hat h}$, corresponding to $l^+_\delta-l^-_\delta+R^+_\delta(\omega)-R^-_\delta(\omega)$ satisfies

$$
\begin{aligned}
|b^{\pm}_{k,\hat{h}}| \leq & C|\ln\delta|^{p+1}\sum_{i,h}\left(\sum_{\alpha=1}^{m}\int_{B_{Ls^{+}_{\delta,\alpha}}(z^{+}_{\alpha})}\Big|\frac{\partial P^{+}_{\delta,Z,i}}{\partial z^{+}_{i,h}}\Big||l^{+}_{\delta}-l^{-}_{\delta}+R^{+}_{\delta}(\omega)-R^{-}_{\delta}(\omega)|\right)\\
&+C|\ln\delta|^{p+1}\sum_{j,\bar{h}}\left(\sum_{\beta=1}^{n}\int_{B_{Ls^{-}_{\delta,\beta}}(z^{-}_{\beta})}\Big|\frac{\partial P^{-}_{\delta,Z,j}}{\partial z^{-}_{j,\bar{h}}}\Big||l^{+}_{\delta}-l^{-}_{\delta}+R^{+}_{\delta}(\omega)-R^{-}_{\delta}(\omega)|\right)\\
\leq & C\varepsilon^{1-\frac{2}{p}}|\ln\varepsilon|^{p}\|l^{+}_{\delta}-l^{-}_{\delta}+R^{+}_{\delta}(\omega)-R^{-}_{\delta}(\omega)\|_{p}.
\end{aligned}
$$

As a result,

$$
\begin{aligned}
&\|Q_{\delta}(l^{+}_{\delta}-l^{-}_{\delta}+R^{+}_{\delta}(\omega)-R^{-}_{\delta}(\omega))\|_{p}\\
\leq & \|l^{+}_{\delta}-l^{-}_{\delta}+R^{+}_{\delta}(\omega)-R^{-}_{\delta}(\omega)\|_{p}+C\sum_{i,h}|b^{+}_{i,h}|\left\|-\delta^{2}\Delta\Big(\frac{\partial P^{+}_{\delta,Z,i}}{\partial z^{+}_{i,h}}\Big)\right\|_{p}\\
&+C\sum_{j,\bar{h}}|b^{-}_{j,\bar{h}}|\left\|-\delta^{2}\Delta\Big(\frac{\partial P^{-}_{\delta,Z,j}}{\partial z^{-}_{j,\bar{h}}}\Big)\right\|_{p}\\
\leq & C\big(\|l^{+}_{\delta}\|_{p}+\|l^{-}_{\delta}\|_{p}+\|R^{+}_{\delta}(\omega)\|_{p}+\|R^{-}_{\delta}(\omega)\|_{p}\big).
\end{aligned}
$$

On the other hand, from Lemma A.1 and (2.13), we can deduce

$$
\begin{aligned}
\|l^{+}_{\delta}\|_{p}= & \left\|\sum_{i=1}^{m}\chi_{\Omega^{+}_{i}}\left(P^{+}_{\delta,Z}-P^{-}_{\delta,Z}-\kappa^{+}_{i}-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)^{p}_{+}-\sum_{i=1}^{m}\left(W_{\delta,z^{+}_{i},a^{+}_{\delta,i}}-a^{+}_{\delta,i}\right)^{p}_{+}\right\|_{p}\\
\leq & \sum_{i=1}^{m}\frac{Cs^{+}_{\delta,i}}{|\ln\varepsilon|}\left\|\left(W_{\delta,z^{+}_{i},a^{+}_{\delta,i}}-a^{+}_{\delta,i}\right)^{p-1}_{+}\right\|_{p}\\
= & O\left(\frac{\delta^{1+\frac{2}{p}}}{|\ln\delta|^{\frac{p-1}{2}+\frac{1}{p}}}\right).
\end{aligned}
$$

For the estimate of $\|R_\delta^+(\omega)\|_p$, we have

$$\begin{aligned}\|R_\delta^+(\omega)\|_\infty =&\Bigg\|\sum_{i=1}^n \chi_{\Omega_i^+}\bigg[\Big(P_{\delta,Z}^+ - P_{\delta,Z}^- + \omega - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\Big)_+^p - \Big(P_{\delta,Z}^+ - P_{\delta,Z}^- - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\Big)_+^p \\ &- p\Big(P_{\delta,Z}^+ - P_{\delta,Z}^- - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\Big)_+^{p-1}\omega\bigg]\Bigg\|_p \\ \le& C\|\omega\|_\infty^2\left\|\sum_{i=1}^n \chi_{\Omega_i^+}\left(P_{\delta,Z}^+ - P_{\delta,Z}^- - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p-2}\right\|_p \\ =& O\left(\frac{\delta^{\frac{2}{p}}\|\omega\|_\infty^2}{|\ln\delta|^{p-3+\frac{1}{p}}}\right).\end{aligned} \tag{3.17}$$

Similarly, we have

$$\|l_\delta^-\|_p = O\left(\frac{\delta^{1+\frac{2}{p}}}{|\ln\delta|^{\frac{p-1}{2}+\frac{1}{p}}}\right), \quad \|R_\delta^-(\omega)\|_p = O\left(\frac{\delta^{\frac{2}{p}}\|\omega\|_\infty^2}{|\ln\delta|^{p-3+\frac{1}{p}}}\right).$$

Thus, we obtain

$$\begin{aligned}\|G_\delta(\omega)\|_\infty \le& \frac{C|\ln\delta|^{\frac{(p-1)^2}{p}}}{\delta^{\frac{2}{p}}}\Big(\|l_\delta^+\|_p + \|l_\delta^-\|_p + \|R_\delta^+(\omega)\|_p + \|R_\delta^-(\omega)\|_p\Big) \\ \le& C|\ln\delta|^{\frac{(p-1)^2}{p}}\left(\frac{\delta}{|\ln\delta|^{\frac{p-1}{2}+\frac{1}{p}}} + \frac{\|\omega\|_\infty^2}{|\ln\delta|^{p-3+\frac{1}{p}}}\right) \\ \le& \delta|\ln\delta|^{\frac{p-1}{2}}\end{aligned} \tag{3.18}$$

Thus, G_δ is a map from M to M.

Step 2. G_δ is a contraction map.

In fact, for any $\omega_i \in M$, $i = 1, 2$, we have

$$G_\delta\omega_1 - G_\delta\omega_2 = (Q_\delta L_\delta)^{-1}Q_\delta\big[R_\delta^+(\omega_1) - R_\delta^+(\omega_2) - (R_\delta^-(\omega_1) - R_\delta^-(\omega_2))\big].$$

Noting that

$$R_\delta^+(\omega_1) = R_\delta^+(\omega_2) = 0, \quad \text{in } \Omega \setminus \cup_{i=1}^m B_{Ls_{\delta,i}^+}(z_i^+),$$

and

$$R_\delta^-(\omega_1) = R_\delta^-(\omega_2) = 0, \quad \text{in } \Omega \setminus \cup_{j=1}^n B_{Ls_{\delta,j}^-}(z_j^-),$$

we can deduce as in Step 1 that

$$\begin{aligned}\|G_\delta\omega_1 - G_\delta\omega_2\|_\infty \leq & \frac{C|\ln\delta|^{\frac{(p-1)^2}{p}}}{\delta^{\frac{2}{p}}}(\|R_\delta^+(\omega_1) - R_\delta^+(\omega_2)\|_p + \|R_\delta^-(\omega_1) - R_\delta^-(\omega_2)\|_p) \\ \leq & C|\ln\delta|^{p-1}\left(\frac{\|\omega_1\|_\infty}{|\ln\delta|^{p-2}} + \frac{\|\omega_2\|_\infty}{|\ln\delta|^{p-2}}\right)\|\omega_1 - \omega_2\|_\infty \\ \leq & C\delta|\ln\delta|^{\frac{p+1}{2}}\|\omega_1 - \omega_2\|_\infty \leq \frac{1}{2}\|\omega_1 - \omega_2\|_\infty.\end{aligned}$$

Combining Step 1 and Step 2, we have proved that G_δ is a contraction map from M to M. By the contraction mapping theorem, there is an unique $\omega_\delta \in M$, such that $\omega_\delta = G_\delta\omega_\delta$. Moreover, it follows from (3.18) that

$$\|\omega_\delta\|_\infty \leq \delta|\ln\delta|^{\frac{p-1}{2}}.$$

□

4. Proof of The main results

In this section, we will choose Z, such that $P^+_{\delta,Z} - P^-_{\delta,Z} + \omega_\delta$, where ω_δ is the map obtained in Proposition 3.3, is a solution of (1.11).

Define

$$\begin{aligned}I(u) = \frac{\delta^2}{2}\int_\Omega |Du|^2 - & \sum_{i=1}^m \frac{1}{p+1}\int_\Omega \chi_{\Omega_i^+}\left(u - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p+1} \\ - & \sum_{j=1}^n \frac{1}{p+1}\int_\Omega \chi_{\Omega_j^-}\left(\frac{2\pi q(x)}{|\ln\varepsilon|} - \kappa_j^- - u\right)_+^{p+1}\end{aligned}$$

and

$$K(Z) = I\left(P^+_{\delta,Z} - P^-_{\delta,Z} + \omega_\delta\right). \tag{4.1}$$

It is well known that if Z is a critical point of $K(Z)$, then $P^+_{\delta,Z} - P^-_{\delta,Z} + \omega_\delta$ is a solution of (1.11). In the following, we will prove that $K(Z)$ has a critical point.

Lemma 4.1. *We have*

$$K(Z) = I\left(P^+_{\delta,Z} - P^-_{\delta,Z}\right) + O\left(\frac{\varepsilon^3}{|\ln\varepsilon|^p}\right).$$

Proof. Recall that $P^+_{\delta,Z}=\sum_{i=1}^m P^+_{\delta,Z,i}, \quad P^-_{\delta,Z}=\sum_{j=1}^n P^-_{\delta,Z,j}$. We have

$$
\begin{aligned}
K(Z)=I\big(P^+_{\delta,Z}-P^-_{\delta,Z}\big)&+\delta^2\int_\Omega D\big(P^+_{\delta,Z}-P^-_{\delta,Z}\big)D\omega_\delta+\frac{\delta^2}{2}\int_\Omega|D\omega_\delta|^2\\
&-\sum_{i=1}^m\frac{1}{p+1}\int_\Omega\chi_{\Omega_i^+}\Bigg[\left(P^+_{\delta,Z}-P^-_{\delta,Z}+\omega_\delta-\kappa_i^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p+1}\\
&-\left(P^+_{\delta,Z}-P^-_{\delta,Z}-\kappa_i^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p+1}\Bigg]\\
&-\sum_{j=1}^n\frac{1}{p+1}\int_\Omega\chi_{\Omega_j^-}\Bigg[\left(P^-_{\delta,Z}-P^+_{\delta,Z}-\omega_\delta-\kappa_j^-+\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p+1}\\
&-\left(P^-_{\delta,Z}-P^+_{\delta,Z}-\kappa_j^-+\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p+1}\Bigg].
\end{aligned}
$$

Using Proposition 3.3 and (3.15), we find

$$
\begin{aligned}
&\int_{\Omega_i^+}\Bigg[\left(P^+_{\delta,Z}-P^-_{\delta,Z}+\omega_\delta-\kappa_i^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p+1}-\left(P^+_{\delta,Z}-P^-_{\delta,Z}-\kappa_i^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p+1}\Bigg]\\
=&\int_{B_{Ls^+_{\delta,i}}(z_i^+)}\Bigg[\left(P^+_{\delta,Z}-P^-_{\delta,Z}+\omega_\delta-\kappa_i^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p+1}-\left(P^+_{\delta,Z}-P^-_{\delta,Z}-\kappa_i^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p+1}\Bigg]\\
=&O\left(\frac{(s^+_{\delta,i})^2\|\omega_\delta\|_\infty}{|\ln\varepsilon|^p}\right)\\
=&O\Big(\frac{\varepsilon^3}{|\ln\varepsilon|^p}\Big).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\delta^2\int_\Omega DP^+_{\delta,Z}D\omega_\delta&=\sum_{i=1}^m\int_\Omega\Big(W_{\delta,z_i^+,a^+_{\delta,i}}-a^+_{\delta,i}\Big)_+^p\omega_\delta\\
=&\sum_{i=1}^m\int_{B_{s^+_{\delta,k}}(z_k^+)}(W_{\delta,z_i^+,a^+_{\delta,i}}-a^+_{\delta,i})_+^p\omega_\delta\\
=&O\Big(\frac{\varepsilon^3}{|\ln\varepsilon|^p}\Big).
\end{aligned}
$$

Next, we estimate $\delta^2 \int_\Omega |D\omega_\delta|^2$. Note that

$$\begin{aligned}
-\delta^2\Delta\omega_\delta =& \sum_{i=1}^m \chi_{\Omega_i^+}\left(P_{\delta,Z}^+ - P_{\delta,Z}^- + \omega_\delta - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p - \sum_{i=1}^m \left(W_{\delta,z_i^+,a_{\delta,i}^+} - a_{\delta,i}^+\right)_+^p \\
& - \sum_{j=1}^n \chi_{\Omega_j^-}\left(P_{\delta,Z}^- - P_{\delta,Z}^+ - \omega_\delta - \kappa_j^- + \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p + \sum_{j=1}^n \left(W_{\delta,z_j^-,a_{\delta,j}^-} - a_{\delta,j}^-\right)_+^p \\
& + \sum_{i=1}^m\sum_{h=1}^2 b_{i,h}^+\left(-\delta^2\Delta\frac{\partial P_{\delta,Z,i}^+}{\partial z_{i,h}^+}\right) + \sum_{j=1}^n\sum_{\bar h=1}^2 b_{j,\bar h}^-\left(-\delta^2\Delta\frac{\partial P_{\delta,Z,j}^-}{\partial z_{j,\bar h}^-}\right).
\end{aligned}$$

Hence, by (2.13)–(2.14), we have

$$\begin{aligned}
\delta^2\int_\Omega |D\omega_\delta|^2 =& \sum_{i=1}^m \int_{\Omega_i^+}\left[\left(P_{\delta,Z}^+ - P_{\delta,Z}^- + \omega_\delta - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p - \left(W_{\delta,z_i^+,a_{\delta,i}^+} - a_{\delta,i}^+\right)_+^p\right]\omega_\delta \\
& - \sum_{j=1}^n \int_{\Omega_j^-}\left[\left(P_{\delta,Z}^- - P_{\delta,Z}^+ - \omega_\delta - \kappa_j^- + \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p - \left(W_{\delta,z_j^-,a_{\delta,j}^-} - a_{\delta,j}^-\right)_+^p\right]\omega_\delta \\
& + \sum_{i=1}^m\sum_{h=1}^2 b_{i,h}^+\int_\Omega\left(-\delta^2\Delta\frac{\partial P_{\delta,Z,i}^+}{\partial z_{i,h}^+}\right)\omega_\delta + \sum_{j=1}^n\sum_{\bar h=1}^2 b_{j,\bar h}^-\int_\Omega\left(-\delta^2\Delta\frac{\partial P_{\delta,Z,j}^-}{\partial z_{j,\bar h}^-}\right)\omega_\delta \\
=& p\sum_{i=1}^m \int_{\Omega_i^+}\left(W_{\delta,z_i^+,a_{\delta,i}^+} - a_{\delta,i}^+\right)_+^{p-1}\left(\frac{s_{\delta,i}^+}{|\ln\varepsilon|} + \omega_\delta\right)\omega_\delta + 0\left(\sum_{i=1}^m\sum_{h=1}^2 \frac{\varepsilon|b_{i,h}^+|\|\omega_\delta\|_\infty}{|\ln\varepsilon|^p}\right) \\
& - p\sum_{j=1}^n \int_{\Omega_j^-}\left(W_{\delta,z_j^-,a_{\delta,j}^-} - a_{\delta,j}^-\right)_+^{p-1}\left(\frac{s_{\delta,j}^-}{|\ln\varepsilon|} + \omega_\delta\right)\omega_\delta + 0\left(\sum_{j=1}^n\sum_{\bar h=1}^2 \frac{\varepsilon|b_{j,\bar h}^-|\|\omega_\delta\|_\infty}{|\ln\varepsilon|^p}\right) \\
=& O\left(\frac{\varepsilon^4}{|\ln\varepsilon|^{p-1}}\right).
\end{aligned}$$

Other terms can be estimated as above. So our assertion follows.

□

Lemma 4.2. *We have*

$$\frac{\partial K(Z)}{\partial z_{i,h}^+} = \frac{\partial}{\partial z_{i,h}^+} I\left(P_{\delta,Z}^+ - P_{\delta,Z}^-\right) + O\Big(\frac{\varepsilon^3}{|\ln\varepsilon|^{p-1}}\Big),\ i = 1,\cdots,m,$$

$$\frac{\partial K(Z)}{\partial z_{j,\bar h}^-} = \frac{\partial}{\partial z_{j,\bar h}^-} I\left(P_{\delta,Z}^+ - P_{\delta,Z}^-\right) + O\Big(\frac{\varepsilon^3}{|\ln\varepsilon|^{p-1}}\Big),\ j = 1,\cdots,n.$$

Proof. We only give the proof of the first estimate.

First, we have

$$
\begin{aligned}
&\frac{\partial K(Z)}{\partial z_{i,h}^+}=\left\langle I'\Big(P_{\delta,Z}^+-P_{\delta,Z}^-+\omega_\delta\Big),\frac{\partial P_{\delta,Z}^+}{\partial z_{i,h}^+}-\frac{\partial P_{\delta,Z}^-}{\partial z_{i,h}^+}+\frac{\partial \omega_\delta}{\partial z_{i,h}^+}\right\rangle\\
=&\frac{\partial}{\partial z_{i,h}^+}I\Big(P_{\delta,Z}^+-P_{\delta,Z}^-\Big)+\left\langle I'\big(P_{\delta,Z}^+-P_{\delta,Z}^-+\omega_\delta\big),\frac{\partial \omega_\delta}{\partial z_{i,h}^+}\right\rangle\\
&-\sum_{k=1}^m\int_{\Omega_k^+}\left[\left(P_{\delta,Z}^+-P_{\delta,Z}^-+\omega_\delta-\kappa_k^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p-\left(P_{\delta,Z}^+-P_{\delta,Z}^--\kappa_k^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p\right]\\
&\qquad\times\left(\frac{\partial P_{\delta,Z}^+}{\partial z_{i,h}^+}-\frac{\partial P_{\delta,Z}^-}{\partial z_{i,h}^+}\right)\\
&-\sum_{l=1}^n\int_{\Omega_l^-}\left[\left(P_{\delta,Z}^--P_{\delta,Z}^+-\omega_\delta-\kappa_l^-+\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p-\left(P_{\delta,Z}^--P_{\delta,Z}^+-\kappa_l^-+\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p\right]\\
&\qquad\times\left(\frac{\partial P_{\delta,Z}^-}{\partial z_{i,h}^+}-\frac{\partial P_{\delta,Z}^+}{\partial z_{i,h}^+}\right).
\end{aligned}
$$

Since $\omega_\delta\in E_{\delta,Z}$, we have

$$
\int_\Omega\Big(W_{\delta,z_k^\pm,a_{\delta,k}^\pm}-a_{\delta,k}^\pm\Big)_+^{p-1}\left(\frac{\partial W_{\delta,z_k^\pm,a_{\delta,k}^\pm}}{\partial z_{k,h}^\pm}-\frac{\partial a_{\delta,k}^\pm}{\partial z_{k,h}^\pm}\right)\omega_\delta=0.
$$

Differentiating the above relation with respect to $z_{i,h}^+$, we can deduce

$$
\begin{aligned}
&\left\langle I'\big(P_{\delta,Z}^+-P_{\delta,Z}^-+\omega_\delta\big),\frac{\partial \omega_\delta}{\partial z_{i,h}^+}\right\rangle\\
=&\sum_{\alpha=1}^m\sum_{\hat h=1}^2 b_{\alpha,\hat h}^+\int_\Omega\left(-\delta^2\Delta\frac{\partial P_{\delta,Z,\alpha}^+}{\partial z_{\alpha,\hat h}^+}\right)\frac{\partial \omega_\delta}{\partial z_{i,h}^+}+\sum_{\beta=1}^n\sum_{\tilde h=1}^2 b_{\beta,\tilde h}^-\int_\Omega\left(-\delta^2\Delta\frac{\partial P_{\delta,Z,\beta}^-}{\partial z_{\beta,\tilde h}^-}\right)\frac{\partial \omega_\delta}{\partial z_{i,h}^+}\\
=&\sum_{\alpha=1}^m\sum_{\hat h=1}^2 p b_{\alpha,\hat h}^+\int_\Omega\Big(W_{\delta,z_\alpha^+,a_{\delta,\alpha}^+}-a_{\delta,\alpha}^+\Big)_+^{p-1}\left(\frac{\partial W_{\delta,z_\alpha^+,a_{\delta,\alpha}^+}}{\partial z_{\alpha,\hat h}^+}-\frac{\partial a_{\delta,\alpha}^+}{\partial z_{\alpha,\hat h}^+}\right)\frac{\partial \omega_\delta}{\partial z_{i,h}^+}\\
&+\sum_{\beta=1}^n\sum_{\tilde h=1}^2 p b_{\beta,\tilde h}^-\int_\Omega\Big(W_{\delta,z_\beta^-,a_{\delta,\beta}^-}-a_{\delta,\beta}^-\Big)_+^{p-1}\left(\frac{\partial W_{\delta,z_\beta^-,a_{\delta,\beta}^-}}{\partial z_{\beta,\tilde h}^-}-\frac{\partial a_{\delta,\beta}^-}{\partial z_{\beta,\tilde h}^-}\right)\frac{\partial \omega_\delta}{\partial z_{i,h}^+}\\
=&O\left(\sum_{\alpha=1}^m\sum_{\hat h=1}^2\frac{\varepsilon|b_{\alpha,\hat h}^+|}{|\ln\varepsilon|^p}+\sum_{\beta=1}^n\sum_{\tilde h=1}^2\frac{\varepsilon|b_{\beta,\tilde h}^-|}{|\ln\varepsilon|^p}\right)=O\left(\frac{\varepsilon^3}{|\ln\varepsilon|^{p-1}}\right).
\end{aligned}
$$

On the other hand, using (3.17) (for the definition of $R_\delta^+(\omega)$, see (3.12)), we obtain

$$
\begin{aligned}
&\sum_{k=1}^{m}\int_{\Omega_k^+}\left[\left(P_{\delta,Z}^{+}-P_{\delta,Z}^{-}+\omega_\delta-\kappa_k^{+}-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p}-\left(P_{\delta,Z}^{+}-P_{\delta,Z}^{-}-\kappa_k^{+}-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p}\right]\frac{\partial P_{\delta,Z,i}^{+}}{\partial z_{i,h}^{+}}\\
=&\sum_{k=1}^{m}\int_{\Omega_k^+}\left[\left(P_{\delta,Z}^{+}-P_{\delta,Z}^{-}+\omega_\delta-\kappa_k^{+}-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p}-\left(P_{\delta,Z}^{+}-P_{\delta,Z}^{-}-\kappa_k^{+}-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p}\right.\\
&\left.-p\left(P_{\delta,Z}^{+}-P_{\delta,Z}^{-}-\kappa_k^{+}-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p-1}\omega_\delta\right]\frac{\partial P_{\delta,Z,i}^{+}}{\partial z_{i,h}^{+}}\\
&+\sum_{k=1}^{m}p\int_{\Omega_k^+}\left[\left(P_{\delta,Z}^{+}-P_{\delta,Z}^{-}-\kappa_k^{+}-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p-1}-\left(W_{\delta,z_k^+,a_{\delta,k}^+}-a_{\delta,k}^{+}\right)_+^{p-1}\right]\frac{\partial P_{\delta,Z,i}^{+}}{\partial z_{i,h}^{+}}\omega_\delta\\
&+O\left(\frac{(s_{\delta,k}^{+})^2\|\omega_\delta\|_\infty}{|\ln\varepsilon|^p}\right)\\
=&\int_\Omega R_\delta^{+}(\omega_\delta)\frac{\partial P_{\delta,Z,i}^{+}}{\partial z_{i,h}^{+}}+\sum_{k=1}^{m}p\int_{\Omega_k^+}\left[\left(P_{\delta,Z}^{+}-P_{\delta,Z}^{-}-\kappa_k^{+}-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p-1}\right.\\
&\left.-\left(W_{\delta,z_k^+,a_{\delta,k}^+}-a_{\delta,k}^{+}\right)_+^{p-1}\right]\frac{\partial P_{\delta,Z,i}^{+}}{\partial z_{i,h}^{+}}\omega_\delta+O\left(\frac{\varepsilon^3}{|\ln\varepsilon|^p}\right)\\
=&O\left(\frac{\varepsilon^3}{|\ln\varepsilon|^{p-1}}\right).
\end{aligned}
$$

In addition, we have

$$
\begin{aligned}
&\int_{\Omega_l^+}\left[\left(P_{\delta,Z}^{+}-P_{\delta,Z}^{-}+\omega_\delta-\kappa_l^{+}-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p}-\left(P_{\delta,Z}^{+}-P_{\delta,Z}^{-}-\kappa_l^{+}-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p}\right]\frac{\partial P_{\delta,Z,i}^{-}}{\partial z_{i,h}^{-}}\\
&=p\int_{\Omega_l^+}\left(P_{\delta,Z}^{+}-P_{\delta,Z}^{-}-\kappa_l^{+}-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p-1}\frac{\partial P_{\delta,Z,i}^{-}}{\partial z_{i,h}^{-}}\omega_\delta\\
&=O\left(\frac{\varepsilon^3}{|\ln\varepsilon|^p}\right).
\end{aligned}
$$

Other teams can be estimated as above. Thus, the estimate follows. $\square$

Proof of Theorem 1.4. Recall that $Z=(Z_m^+,Z_n^-)$. Set

$$\begin{aligned}\Phi(Z_m^+,Z_n^-)=&\sum_{i=1}^m 4\pi^2\kappa_i^+q(z_i^+)-\sum_{j=1}^n 4\pi^2\kappa_j^-q(z_j^-)+\sum_{i=1}^m\pi(\kappa_i^+)^2g(z_i^+,z_i^+)\\&+\sum_{j=1}^n\pi(\kappa_j^-)^2g(z_j^-,z_j^-)-\sum_{i\neq k}\pi\kappa_i^+\kappa_k^+\bar{G}(z_i^+,z_k^+)-\sum_{j\neq l}\pi\kappa_j^-\kappa_l^-\bar{G}(z_l^-,z_j^-)\\&+\sum_{i=1}^m\sum_{j=1}^n 2\pi\kappa_i^+\kappa_j^-\bar{G}(z_i^+,z_j^-).\end{aligned}$$

Note that the Kirchhoff–Routh function associated to the vortex dynamics now is

$$\begin{aligned}\mathcal{W}(Z_m^+,Z_n^-)=&\frac{1}{2}\sum_{i,k=1,i\neq k}^m\kappa_i^+\kappa_k^+G(z_i^+,z_k^+)+\frac{1}{2}\sum_{j,l=1,j\neq l}^n\kappa_j^-\kappa_l^-G(z_j^-,z_l^-)\\&+\frac{1}{2}\sum_{i=1}^m(\kappa_i^+)^2H(z_i^+,z_i^+)+\frac{1}{2}\sum_{j=1}^n(\kappa_j^-)^2H(z_j^-,z_j^-)\\&-\sum_{i=1}^m\sum_{j=1}^n\kappa_i^+\kappa_j^-G(z_i^+,z_j^-)+\sum_{i=1}^m\kappa_i^+\psi_0(z_i^+)-\sum_{j=1}^n\kappa_j^-\psi_0(z_j^-).\end{aligned}$$

Recall that $h(z_i,z_j)=-H(z_i,z_j)$, it is easy to check that

$$\Phi(Z_m^+,Z_n^-)=-4\pi^2\mathcal{W}(Z_m^+,Z_n^-)+\pi\ln R\left(\sum_{i=1}^m(\kappa_i^+)^2+\sum_{j=1}^n(\kappa_j^-)^2\right).$$

Hence, $\Phi(Z_m^+,Z_N^-)$ and $\mathcal{W}(Z_m^+,Z_N^-)$ possess the same critical points.

By Lemma 4.1, 4.2 and Proposition A.2, A.3, we have

$$K(Z)=\frac{C\delta^2}{\ln\frac{R}{\varepsilon}}+\frac{\pi(p-1)\delta^2}{4(\ln\frac{R}{\varepsilon})^2}\left(\sum_{i=1}^m(\kappa_i^+)^2+\sum_{j=1}^n(\kappa_j^-)^2\right)+\frac{\delta^2}{|\ln\varepsilon|^2}\Phi(Z)+0\left(\frac{\delta^2\ln|\ln\varepsilon|}{|\ln\varepsilon|^3}\right)$$

and

$$\frac{\partial K(Z)}{\partial z_{i,h}^\pm}=\frac{\delta^2}{|\ln\varepsilon|^2}\frac{\partial\Phi(Z)}{\partial z_{i,h}^\pm}+O\left(\frac{\delta^2\ln|\ln\varepsilon|}{|\ln\varepsilon|^3}\right).$$

Thus, the existence of a C^1-stable critical point of Kirchhoff-Routh function $\mathcal{W}(Z)$ implies that $K(Z)$ has a critical point.

Thus we get a solution w_δ for (1.11). Let $u_\varepsilon=\frac{|\ln\varepsilon|}{2\pi}w_\delta$, $\delta=\varepsilon\left(\frac{|\ln\varepsilon|}{2\pi}\right)^{\frac{1-p}{2}}$, it is not difficult to check that u_ε has all the properties listed in Theorem 1.4 and thus the proof of Theorem 1.4 is complete. □

Now we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. By Theorem 1.4, we obtain that u_ε is a solution to (1.10).

Set

$$\begin{aligned}
&\mathbf{v}_\varepsilon=(\nabla(u_\varepsilon-q))^\perp,\qquad \omega_\varepsilon=\nabla\times\mathbf{v}_\varepsilon,\\
P_\varepsilon=&\sum_{i=1}^m\frac{1}{p+1}\chi_{\Omega_i^+}\left(u_\varepsilon-q-\frac{\kappa_i^+|\ln\varepsilon|}{2\pi}\right)_+^{p+1}\\
&+\sum_{j=1}^n\frac{1}{p+1}\chi_{\Omega_j^-}\left(q-\frac{\kappa_j^-|\ln\varepsilon|}{2\pi}-u_\varepsilon\right)_+^{p+1}-\frac12|\nabla(u_\varepsilon-q)|^2.
\end{aligned}$$

Then $(\mathbf{v}_\varepsilon,P_\varepsilon)$ forms a stationary solution for problem (1.1).

We now just need to verify as $\varepsilon\to0$

$$\int_\Omega\omega_\varepsilon\to\sum_{j=1}^m\kappa_j^+-\sum_{j=1}^n\kappa_j^-.$$

By direct calculations, we find that

$$\begin{aligned}
\int_\Omega\omega_\varepsilon&=\sum_{i=1}^m\frac{1}{\varepsilon^2}\int_\Omega\chi_{\Omega_i^+}\left(u_\varepsilon-q-\frac{\kappa_i^+|\ln\varepsilon|}{2\pi}\right)_+^p-\sum_{j=1}^n\frac{1}{\varepsilon^2}\int_\Omega\chi_{\Omega_j^-}\left(q-\frac{\kappa_j^-|\ln\varepsilon|}{2\pi}-u_\varepsilon\right)_+^p\\
&=\sum_{i=1}^m\frac{|\ln\varepsilon|^p}{(2\pi)^p\varepsilon^2}\int_{\Omega_i^+}\left(w_\delta-\kappa_i^+-\frac{2\pi q}{|\ln\varepsilon|}\right)_+^p-\sum_{j=1}^n\frac{|\ln\varepsilon|^p}{(2\pi)^p\varepsilon^2}\int_{\Omega_j^-}\left(\frac{2\pi q}{|\ln\varepsilon|}-\kappa_j^--w_\delta\right)_+^p\\
&=\frac{|\ln\varepsilon|^p}{(2\pi)^p\varepsilon^2}\sum_{i=1}^m\int_{B_{Ls_{\delta,i}^+}(z_i^+)}\left(W_{\delta,z_i^+,a_{\delta,i}^+}-a_{\delta,i}^++O\Big(\frac{s_{\delta,i}^+}{|\ln\varepsilon|}\Big)\right)_+^p\\
&\quad-\frac{|\ln\varepsilon|^p}{(2\pi)^p\varepsilon^2}\sum_{j=1}^n\int_{B_{Ls_{\delta,j}^-}(z_j^-)}\left(W_{\delta,z_j^-,a_{\delta,j}^-}-a_{\delta,j}^-+O\Big(\frac{s_{\delta,j}^-}{|\ln\varepsilon|}\Big)\right)_+^p\\
&=\sum_{i=1}^m\frac{(s_{\delta,i}^+)^2|\ln\varepsilon|^p}{(2\pi)^p\varepsilon^2}\left(\frac{\delta}{s_{\delta,i}^+}\right)^{\frac{2p}{p-1}}\int_{B_1(0)}\phi^p\\
&\quad-\sum_{j=1}^n\frac{(s_{\delta,j}^-)^2|\ln\varepsilon|^p}{(2\pi)^p\varepsilon^2}\left(\frac{\delta}{s_{\delta,j}^-}\right)^{\frac{2p}{p-1}}\int_{B_1(0)}\phi^p+o(1)\\
&=\sum_{i=1}^m\frac{a_{\delta,i}^+|\ln\varepsilon|}{\ln\frac{R}{s_{\delta,i}^+}}-\sum_{j=1}^n\frac{a_{\delta,j}^-|\ln\varepsilon|}{\ln\frac{R}{s_{\delta,j}^-}}+o(1)\\
&\to\sum_{j=1}^m\kappa_j^+-\sum_{j=1}^n\kappa_j^-,\quad\text{as }\varepsilon\to0.
\end{aligned}$$

Therefore, the result follows. □

Remark 4.3. To regularize pairs of vortices with equi-strength κ, we do not need $\chi_{\Omega_i^+}$ and $\chi_{\Omega_j^-}$, that is, we only need to consider the following problem

$$\begin{cases}-\varepsilon^2\Delta u=(u-q-\frac{\kappa}{2\pi}\ln\frac{1}{\varepsilon})_+^p-(q-\frac{\kappa}{2\pi}\ln\frac{1}{\varepsilon}-u)_+^p, & x\in\Omega,\\ u=0, & x\in\partial\Omega.\end{cases}$$

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Appendix A. Energy expansion

In this section we will give precise expansions of $I\left(P_{\delta,Z}^+-P_{\delta,Z}^-\right)$ and $\frac{\partial}{\partial z_{i,h}^{\pm}}I\left(P_{\delta,Z}^+-P_{\delta,Z}^-\right)$, which have been used in section 4.

We always assume that $z_i^+,z_j^-\in\Omega$ satisfies

$$\begin{aligned}&d(z_i^+,\partial\Omega)\ge\varrho,\ d(z_j^-,\partial\Omega)\ge\varrho,\quad |z_i^+-z_k^+|\ge\varrho^{\bar L},\quad i,k=1,\cdots,m,\ i\ne k\\ &|z_j^--z_l^-|\ge\varrho^{\bar L},\quad |z_i^+-z_j^-|\ge\varrho^{\bar L},\quad j,l=1,\cdots,n,\ j\ne l,\end{aligned}$$

where $\varrho>0$ is a fixed small constant and $\bar L>0$ is a fixed large constant.

Lemma A.1. *For $x\in\Omega_i^+$, $i=1,2,\cdots,m$ and $x\in\Omega_j^-$, $j=1,2,\cdots,m$, we have*

$$\begin{aligned}&P_{\delta,Z}^+(x)-P_{\delta,Z}^-(x)>\kappa_i^++\frac{2\pi q(x)}{|\ln\varepsilon|},\quad x\in B_{s_{\delta,i}^+(1-Ts_{\delta,i}^+)}(z_i^+),\\ &P_{\delta,Z}^-(x)-P_{\delta,Z}^+(x)>\kappa_j^--\frac{2\pi q(x)}{|\ln\varepsilon|},\quad x\in B_{s_{\delta,j}^-(1-Ts_{\delta,j}^-)}(z_j^-),\end{aligned}$$

where $T>0$ is a large constant; while

$$\begin{aligned}&P_{\delta,Z}^+(x)-P_{\delta,Z}^-(x)<\kappa_i^++\frac{2\pi q(x)}{|\ln\varepsilon|},\quad x\in\Omega_i^+\setminus B_{s_{\delta,i}^+(1+(s_{\delta,i}^+)^\sigma)}(z_i^+),\\ &P_{\delta,Z}^-(x)-P_{\delta,Z}^+(x)<\kappa_j^--\frac{2\pi q(x)}{|\ln\varepsilon|},\quad x\in\Omega_j^-\setminus B_{s_{\delta,j}^-(1+(s_{\delta,j}^-)^\sigma)}(z_j^-),\end{aligned}$$

where $\sigma>0$ is a small constant.

Proof. The proof is exactly same as Lemma A.1 in [13]. For reader's convenience, we give the proof for $P_{\delta,Z}^+-P_{\delta,Z}^-$ here.

Suppose that $x\in B_{s_{\delta,i}^+(1-Ts_{\delta,i}^+)}(z_i^+)$. It follows from (2.13) and $\phi_1'(s)<0$ that

$$\begin{aligned}&P_{\delta,Z}^+(x)-P_{\delta,Z}^-(x)-\kappa_i^+-\frac{2\pi q(x)}{|\ln\varepsilon|}=W_{\delta,z_i^+,a_{\delta,i}^+}(x)-a_{\delta,i}^++O\left(\frac{s_{\delta,i}^+}{|\ln\varepsilon|}\right)\\ =&\frac{a_{\delta,i}^+}{|\phi'(1)||\ln\frac{R}{s_{\delta,i}^+}|}\phi\Big(\frac{|x-z_i^+|}{s_{\delta,i}^+}\Big)+O\Big(\frac{\varepsilon}{|\ln\varepsilon|}\Big)>0,\end{aligned}$$

if $T>0$ is large. On the other hand, if $x\in\Omega_i^+\setminus B_{(s_{\delta,i}^+)^{\tilde\sigma}}(z_i^+)$, where $\tilde\sigma>\sigma>0$ is a fixed small constant, then

$$\begin{aligned}
&P_{\delta,Z}^+(x)-P_{\delta,Z}^-(x)-\kappa_i^+-\frac{2\pi q(x)}{\kappa|\ln\varepsilon|}\\
\le&\sum_{i=1}^m a_{\delta,i}^+\ln\frac{R}{|x-z_i^+|}/\ln\frac{R}{s_{\delta,i}^+}-\kappa_i^+-\frac{2\pi q(x)}{\kappa|\ln\varepsilon|}+o(1)\\
\le&C\tilde\sigma-\kappa_i^++o(1)<0.
\end{aligned}$$

Finally, if $x\in B_{(s_{\delta,i}^+)^{\tilde\sigma}}(z_i^+)\setminus B_{s_{\delta,i}^+(1+T(s_{\delta,i}^+)^{\tilde\sigma})}(z_i^+)$ for some i, then

$$\begin{aligned}
&P_{\delta,Z}^+(x)-P_{\delta,Z}^-(x)-\kappa_i^+-\frac{2\pi q(x)}{\kappa|\ln\varepsilon|}=W_{\delta,z_i^+,a_{\delta,i}^+}(x)-a_{\delta,i}^++O\left(\frac{(s_{\delta,i}^+)^{\tilde\sigma}}{\ln\frac{R}{s_{\delta,i}^+}}\right)\\
=&a_{\delta,i}^+\frac{\ln\frac{R}{|x-z_i^+|}}{\ln\frac{R}{s_{\delta,i}^+}}-a_{\delta,i}^++O\left(\frac{(s_{\delta,i}^+)^{\tilde\sigma}}{\ln\frac{R}{s_{\delta,i}^+}}\right)\\
\le&-a_{\delta,i}^+\frac{\ln(1+T(s_{\delta,i}^+)^{\tilde\sigma})}{\ln\frac{R}{s_{\delta,i}^+}}+O\left(\frac{(s_{\delta,i}^+)^{\tilde\sigma}}{\ln\frac{R}{s_{\delta,i}^+}}\right)<0,
\end{aligned}$$

if $T>0$ is large. Note that by the choice of $\tilde\sigma$, $B_{s_{\delta,i}^+(1+(s_{\delta,i}^+)^{\sigma})}(z_i^+)\supset B_{s_{\delta,i}^+(1+T(s_{\delta,i}^+)^{\tilde\sigma})}(z_i^+)$ for small δ. We therefore derive our conclusion. □

Proposition A.2. *We have*

$$\begin{aligned}
I\left(P_{\delta,Z}^+-P_{\delta,Z}^-\right)=&\frac{C\delta^2}{\ln\frac{R}{\varepsilon}}+\frac{\pi(p-1)\delta^2}{4(\ln\frac{R}{\varepsilon})^2}\left(\sum_{i=1}^m(\kappa_i^+)^2+\sum_{j=1}^n(\kappa_j^-)^2\right)+\sum_{i=1}^m\frac{4\pi^2\delta^2\kappa_i^+q(z_i^+)}{|\ln\varepsilon||\ln\frac{R}{\varepsilon}|}\\
&-\sum_{j=1}^n\frac{4\pi^2\delta^2\kappa_j^-q(z_j^-)}{|\ln\varepsilon||\ln\frac{R}{\varepsilon}|}+\sum_{i=1}^m\frac{\pi\delta^2(\kappa_i^+)^2g(z_i^+,z_i^+)}{(\ln\frac{R}{\varepsilon})^2}+\sum_{j=1}^n\frac{\pi\delta^2(\kappa_j^-)^2g(z_j^-,z_j^-)}{(\ln\frac{R}{\varepsilon})^2}\\
&-\sum_{k\neq i}^m\frac{\pi\delta^2\kappa_i^+\kappa_k^+\bar G(z_k^+,z_i^+)}{(\ln\frac{R}{\varepsilon})^2}-\sum_{l\neq j}^n\frac{\pi\delta^2\kappa_j^-\kappa_l^-\bar G(z_l^-,z_j^-)}{(\ln\frac{R}{\varepsilon})^2}\\
&+\sum_{i=1}^m\sum_{j=1}^n\frac{2\pi\delta^2\kappa_i^+\kappa_j^-\bar G(z_i^+,z_j^-)}{(\ln\frac{R}{\varepsilon})^2}+O\left(\frac{\delta^2\ln|\ln\varepsilon|}{|\ln\varepsilon|^3}\right).
\end{aligned}$$

where C is a positive constant.

Proof. Taking advantage of (2.3), we have

$$
\begin{aligned}
\delta^2\int_\Omega \big|D(P^+_{\delta,Z}-P^-_{\delta,Z})\big|^2 =& \sum_{k=1}^m\sum_{i=1}^m\int_\Omega \big(W_{\delta,z_k^+,a^+_{\delta,k}}-a^+_{\delta,k}\big)^p_+P^+_{\delta,Z,i}\\
&+\sum_{l=1}^n\sum_{j=1}^n\int_\Omega\big(W_{\delta,z_l^-,a^-_{\delta,l}}-a^-_{\delta,l}\big)^p_+P^-_{\delta,Z,j}-2\sum_{j=1}^n\sum_{i=1}^m\int_\Omega\big(W_{\delta,z_i^+,a^+_{\delta,i}}-a^+_{\delta,i}\big)^p_+P^-_{\delta,Z,j}.
\end{aligned}
$$

First, we estimate

$$
\begin{aligned}
&\int_{B_{s^+_{\delta,i}}(z_i^+)}\Big(W_{\delta,z_i^+,a^+_{\delta,i}}-a^+_{\delta,i}\Big)^p_+\left(W_{\delta,z_i^+,a^+_{\delta,i}}-\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}g(x,z_i^+)\right)\\
=&\int_{B_{s^+_{\delta,i}}(z_i^+)}\big(W_{\delta,z_i^+,a^+_{\delta,i}}-a^+_{\delta,i}\big)^{p+1}+a^+_{\delta,i}\int_{B_{s^+_{\delta,i}}(z_i^+)}\big(W_{\delta,z_i^+,a^+_{\delta,i}}-a^+_{\delta,i}\big)^p\\
&-\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}\int_{B_{s^+_{\delta,i}}(z_i)}\big(W_{\delta,z_i^+,a^+_{\delta,i}}-a^+_{\delta,i}\big)^p g(x,z_i^+)\\
=&\Big(\frac{\delta}{s^+_{\delta,i}}\Big)^{\frac{2(p+1)}{p-1}}(s^+_{\delta,i})^2\int_{B_1(0)}\phi^{p+1}+a^+_{\delta,i}\Big(\frac{\delta}{s^+_{\delta,i}}\Big)^{\frac{2p}{p-1}}(s^+_{\delta,i})^2\int_{B_1(0)}\phi^p\\
&-\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}\Big(\frac{\delta}{s^+_{\delta,i}}\Big)^{\frac{2p}{p-1}}g(z_i^+,z_i^+)(s^+_{\delta,i})^2\int_{B_1(0)}\phi^p+O\left(\frac{(s^+_{\delta,i})^3}{|\ln\varepsilon|^{p+1}}\right)\\
=&\frac{\pi(p+1)}{2}\frac{\delta^2(a^+_{\delta,i})^2}{(\ln\frac{R}{s^+_{\delta,i}})^2}+\frac{2\pi\delta^2(a^+_{\delta,i})^2}{\ln\frac{R}{s^+_{\delta,i}}}-\frac{2\pi\delta^2(a^+_{\delta,i})^2}{(\ln\frac{R}{s^+_{\delta,i}})^2}g(z_i^+,z_i^+)+O\left(\frac{\varepsilon^3}{|\ln\varepsilon|^{p+1}}\right).
\end{aligned}
$$

Next, for $k\neq i$,

$$
\begin{aligned}
&\int_{B_{s^+_{\delta,k}}(z_k^+)}\big(W_{\delta,z_k^+,a^+_{\delta,k}}-a^+_{\delta,k}\big)^p_+\left(W_{\delta,z_i^+,a^+_{\delta,i}}-\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}g(x,z_i^+)\right)\\
=&\Big(\frac{\delta}{s^+_{\delta,k}}\Big)^{\frac{2p}{p-1}}\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}\int_{B_{s^+_{\delta,k}}(z_k^+)}\phi^p\Big(\frac{|x-z_k^+|}{s^+_{\delta,k}}\Big)\bar G(x,z_i^+)\\
=&\Big(\frac{\delta}{s^+_{\delta,k}}\Big)^{\frac{2p}{p-1}}\frac{a^+_{\delta,i}(s^+_{\delta,k})^2}{\ln\frac{R}{s^+_{\delta,i}}}\bar G(z_k^+,z_i^+)\int_{B_1(0)}\phi^p+O\left(\frac{(s^+_{\delta,k})^3}{|\ln\varepsilon|^{p+1}}\right)\\
=&\frac{2\pi\delta^2a^+_{\delta,i}a^+_{\delta,k}}{|\ln\frac{R}{s^+_{\delta,i}}||\ln\frac{R}{s^+_{\delta,k}}|}\bar G(z_i^+,z_k^+)+O\left(\frac{\varepsilon^3}{|\ln\varepsilon|^{p+1}}\right).
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&\int_{B_{s^+_{\delta,i}}(z_i^+)}\big(W_{\delta,z_i^+,a^+_{\delta,i}}-a^+_{\delta,i}\big)^p_+\left(W_{\delta,z_j^-,a^-_{\delta,j}}-\frac{a^-_{\delta,j}}{\ln\frac{R}{s^-_{\delta,j}}}g(x,z_j^-)\right)\\
=&\Big(\frac{\delta}{s^+_{\delta,i}}\Big)^{\frac{2p}{p-1}}\frac{a^-_{\delta,j}}{\ln\frac{R}{s^-_{\delta,j}}}\int_{B_{s^+_{\delta,i}}(z_i^+)}\phi^p\Big(\frac{|x-z_i^+|}{s^+_{\delta,i}}\Big)\bar G(x,z_j^-)\\
=&\Big(\frac{\delta}{s^+_{\delta,i}}\Big)^{\frac{2p}{p-1}}\frac{a^-_{\delta,j}(s^+_{\delta,i})^2}{\ln\frac{R}{s^-_{\delta,j}}}\bar G(z_j^-,z_i^+)\int_{B_1(0)}\phi^p+O\left(\frac{(s^+_{\delta,i})^3}{|\ln\varepsilon|^{p+1}}\right)\\
=&\frac{2\pi\delta^2a^+_{\delta,i}a^-_{\delta,j}}{|\ln\frac{R}{s^+_{\delta,i}}||\ln\frac{R}{s^-_{\delta,j}}|}\bar G(z_i^+,z_j^-)+O\left(\frac{\varepsilon^3}{|\ln\varepsilon|^{p+1}}\right).
\end{aligned}
$$

By Lemma A.1 and (2.13),

$$
\begin{aligned}
&\sum_{k=1}^m\int_\Omega\chi_{\Omega_k^+}\left(P^+_{\delta,Z}-P^-_{\delta,Z}-\kappa_k^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)^{p+1}_+\\
=&\sum_{k=1}^m\int_{B_{Ls^+_{\delta,k}}(z_k^+)}\left(P^+_{\delta,Z}-P^-_{\delta,Z}-\kappa_k^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)^{p+1}_+\\
=&\sum_{k=1}^m\int_{B_{Ls^+_{\delta,k}}(z_k^+)}\left(W_{\delta,z_k^+,a^+_{\delta,k}}-a^+_{\delta,k}+O\Big(\frac{s^+_{\delta,k}}{|\ln\varepsilon|}\Big)\right)^{p+1}_+\\
=&\sum_{k=1}^m\left(\frac{\delta}{s^+_{\delta,k}}\right)^{\frac{2(p+1)}{p-1}}\int_{B_{s^+_{\delta,k}}(z_k^+)}\phi^{p+1}\Big(\frac{|x-z_k^+|}{s^+_{\delta,k}}\Big)+O\left(\frac{(s^+_{\delta,k})^3}{|\ln\varepsilon|^{p+1}}\right)\\
=&\sum_{k=1}^m\left(\frac{\delta}{s^+_{\delta,k}}\right)^{\frac{2(p+1)}{p-1}}(s^+_{\delta,k})^2\int_{B_1(0)}\phi^{p+1}+O\left(\frac{(s^+_{\delta,k})^3}{|\ln\varepsilon|^{p+1}}\right)\\
=&\sum_{k=1}^m\frac{\pi(p+1)}{2}\frac{\delta^2(a^+_{\delta,k})^2}{(\ln\frac{R}{s^+_{\delta,k}})^2}+O\left(\frac{\varepsilon^3}{|\ln\varepsilon|^{p+1}}\right).
\end{aligned}
$$

Other terms can be estimated as above. So, we have proved

$$
\begin{aligned}
I\left(P^+_{\delta,Z}-P^-_{\delta,Z}\right)=&\sum_{i=1}^{m}\left[\frac{\pi(p+1)}{4}\frac{\delta^2(a^+_{\delta,i})^2}{|\ln\frac{R}{s^+_{\delta,i}}|^2}+\frac{\pi\delta^2(a^+_{\delta,i})^2}{|\ln\frac{R}{s^+_{\delta,i}}|}-\frac{\pi g(z^+_i,z^+_i)\delta^2(a^+_{\delta,i})^2}{|\ln\frac{R}{s^+_{\delta,i}}|^2}\right]\\
&+\sum_{j=1}^{n}\left[\frac{\pi(p+1)}{4}\frac{\delta^2(a^-_{\delta,j})^2}{|\ln\frac{R}{s^-_{\delta,j}}|^2}+\frac{\pi\delta^2(a^-_{\delta,j})^2}{|\ln\frac{R}{s^-_{\delta,j}}|}-\frac{\pi g(z^-_j,z^-_j)\delta^2(a^-_{\delta,j})^2}{|\ln\frac{R}{s^-_{\delta,j}}|^2}\right]\\
&+\sum_{k\neq i}^{m}\frac{\pi\bar G(z^+_k,z^+_i)\delta^2a^+_{\delta,i}a^+_{\delta,k}}{|\ln\frac{R}{s^+_{\delta,i}}||\ln\frac{R}{s^+_{\delta,k}}|}+\sum_{l\neq j}^{n}\frac{\pi\bar G(z^-_l,z^-_j)\delta^2a^-_{\delta,l}a^-_{\delta,j}}{|\ln\frac{R}{s^-_{\delta,l}}||\ln\frac{R}{s^-_{\delta,j}}|}\\
&-\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{2\pi\bar G(z^+_i,z^-_j)\delta^2a^+_{\delta,i}a^-_{\delta,j}}{|\ln\frac{R}{s^+_{\delta,i}}||\ln\frac{R}{s^-_{\delta,j}}|}-\frac{\pi\delta^2}{2}\left(\sum_{i=1}^{m}\frac{(a^+_{\delta,i})^2}{|\ln\frac{R}{s^+_{\delta,i}}|^2}\right)\\
&-\frac{\pi\delta^2}{2}\left(\sum_{j=1}^{n}\frac{(a^-_{\delta,j})^2}{|\ln\frac{R}{s^-_{\delta,j}}|^2}\right)+O\left(\frac{\varepsilon^3}{|\ln\varepsilon|^{p+1}}\right).
\end{aligned}
$$

Thus, the result follows from Remark 2.2. □

Proposition A.3. *We have*

$$
\begin{aligned}
\frac{\partial}{\partial z^+_{i,h}}I\left(P^+_{\delta,Z}-P^-_{\delta,Z}\right)=&\frac{4\pi^2\delta^2\kappa^+_i}{|\ln\varepsilon||\ln\frac{R}{\varepsilon}|}\frac{\partial q(z^+_i)}{\partial z^+_{i,h}}+\frac{2\pi\delta^2(\kappa^+_i)^2}{(\ln\frac{R}{\varepsilon})^2}\frac{\partial g(z^+_i,z^+_i)}{\partial z^+_{i,h}}\\
&-\sum_{k\neq i}^{m}\frac{2\pi\delta^2\kappa^+_i\kappa^+_k}{(\ln\frac{R}{\varepsilon})^2}\frac{\partial\bar G(z^+_k,z^+_i)}{\partial z^+_{i,h}}+\sum_{l=1}^{n}\frac{2\pi\delta^2\kappa^+_i\kappa^-_l}{(\ln\frac{R}{\varepsilon})^2}\frac{\partial\bar G(z^+_i,z^-_l)}{\partial z^+_{i,h}}+O\left(\frac{\delta^2\ln|\ln\varepsilon|}{|\ln\varepsilon|^3}\right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial z^-_{j,\bar h}}I\left(P^+_{\delta,Z}-P^-_{\delta,Z}\right)=&-\frac{4\pi^2\delta^2\kappa^-_j}{|\ln\varepsilon||\ln\frac{R}{\varepsilon}|}\frac{\partial q(z^-_j)}{\partial z^-_{j,\bar h}}+\frac{2\pi\delta^2(\kappa^-_j)^2}{(\ln\frac{R}{\varepsilon})^2}\frac{\partial g(z^-_j,z^-_j)}{\partial z^-_{j,\bar h}}\\
&-\sum_{l\neq j}^{n}\frac{2\pi\delta^2\kappa^-_j\kappa^-_l}{(\ln\frac{R}{\varepsilon})^2}\frac{\partial\bar G(z^-_l,z^-_j)}{\partial z^-_{j,\bar h}}+\sum_{k=1}^{m}\frac{2\pi\delta^2\kappa^-_j\kappa^+_k}{(\ln\frac{R}{\varepsilon})^2}\frac{\partial\bar G(z^-_j,z^+_k)}{\partial z^-_{j,\bar h}}+O\left(\frac{\delta^2\ln|\ln\varepsilon|}{|\ln\varepsilon|^3}\right).
\end{aligned}
$$

Proof. Direct computation yields that

$$
\begin{aligned}
&\frac{\partial}{\partial z_{i,h}^+} I\left(P_{\delta,Z}^+ - P_{\delta,Z}^-\right)\\
&\quad= \sum_{k=1}^{m} \int_{B_{Ls_{\delta,k}^+}(z_k^+)} \left[\left(W_{\delta,z_k^+,a_{\delta,k}^+} - a_{\delta,k}^+\right)_+^p - \left(P_{\delta,Z}^+ - P_{\delta,Z}^- - \kappa_k^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p\right] \frac{\partial P_{\delta,Z}^+}{\partial z_{i,h}^+}\\
&\qquad+ \sum_{l=1}^{n} \int_{B_{Ls_{\delta,l}^-}(z_l^-)} \left[\left(W_{\delta,z_l^-,a_{\delta,l}^-} - a_{\delta,l}^-\right)_+^p - \left(P_{\delta,Z}^- - P_{\delta,Z}^+ - \kappa_l^- + \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p\right] \frac{\partial P_{\delta,Z}^-}{\partial z_{i,h}^+}\\
&\qquad- \sum_{k=1}^{m} \int_{B_{Ls_{\delta,k}^+}(z_k^+)} \left[\left(W_{\delta,z_k^+,a_{\delta,k}^+} - a_{\delta,k}^+\right)_+^p - \left(P_{\delta,Z}^+ - P_{\delta,Z}^- - \kappa_k^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p\right] \frac{\partial P_{\delta,Z}^-}{\partial z_{i,h}^+}\\
&\qquad- \sum_{l=1}^{n} \int_{B_{Ls_{\delta,l}^-}(z_l^-)} \left[\left(W_{\delta,z_l^-,a_{\delta,l}^-} - a_{\delta,l}^-\right)_+^p - \left(P_{\delta,Z}^- - P_{\delta,Z}^+ - \kappa_l^+ + \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p\right] \frac{\partial P_{\delta,Z}^+}{\partial z_{i,h}^+}.
\end{aligned}
$$

For $k \neq i$, from (2.13), we have

$$
\begin{aligned}
&\int_{B_{Ls_{\delta,k}^+}(z_k^+)} \left[(W_{\delta,z_k^+,a_{\delta,k}^+} - a_{\delta,k}^+)_+^p - \left(P_{\delta,Z}^+ - P_{\delta,Z}^- - \kappa_k^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p\right] \frac{\partial P_{\delta,Z,i}^+}{\partial z_{i,h}^+}\\
=&\int_{B_{Ls_{\delta,k}^+}(z_k^+)} \left[\left(W_{\delta,z_k^+,a_{\delta,k}^+} - a_{\delta,k}^+\right)^{p-1} \frac{s_{\delta,k}^+}{|\ln\varepsilon|}\right] \frac{C}{\ln\frac{R}{s_{\delta,i}^+}}\\
=&O\left(\frac{\varepsilon^3}{|\ln\varepsilon|^{p+1}}\right).
\end{aligned}
$$

Using (2.13), Lemma A.1 and Remark 2.2, we find that

$$
\begin{aligned}
&\int_{B_{Ls^+_{\delta,i}}(z_i^+)} \Big[\Big(W_{\delta,z_i^+,a^+_{\delta,i}} - a^+_{\delta,i}\Big)_+^p - \Big(P^+_{\delta,Z} - P^-_{\delta,Z} - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\Big)_+^p\Big] \frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}} \\
=&\int_{B_{s^+_{\delta,i}(1+(s^+_{\delta,i})^\sigma)}(z_i)} \Big[\big(W_{\delta,z_i^+,a^+_{\delta,i}} - a^+_{\delta,i}\big)_+^p - \Big(P^+_{\delta,Z} - P^-_{\delta,Z} - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\Big)_+^p\Big] \frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}} \\
=&p\int_{B_{s^+_{\delta,i}}(z_i^+)} \big(W_{\delta,z_i^+,a^+_{\delta,i}} - a^+_{\delta,i}\big)_+^{p-1} \Big[\frac{2\pi}{|\ln\varepsilon|}\big\langle Dq(z_i^+), x - z_i^+\big\rangle + \frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}\big\langle Dg(z_i^+,z_i^+), x - z_i^+\big\rangle \\
&- \sum_{k\neq i}^m \frac{a^+_{\delta,k}}{\ln\frac{R}{s^+_{\delta,k}}}\big\langle D\bar{G}(z_i^+,z_k^+), x - z_i^+\big\rangle + \sum_{l=1}^n \frac{a^-_{\delta,l}}{\ln\frac{R}{s^-_{\delta,l}}}\big\langle D\bar{G}(z_i^+,z_l^-), x - z_i^+\big\rangle\Big] \frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}} \\
&+ O\Big(\frac{\varepsilon^{2+\sigma}}{|\ln\varepsilon|^{p+1}}\Big) \\
=&-\frac{p\delta^2 a^+_{\delta,i}}{|\phi'(1)||\ln\frac{R}{s^+_{\delta,i}}|}\Big(\frac{2\pi}{|\ln\varepsilon|}\frac{\partial q(z_i^+)}{\partial z^+_{i,h}} + \frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}\frac{\partial g(z_i^+,z_i^+)}{\partial z^+_{i,h}} - \sum_{k\neq i}^m \frac{a^+_{\delta,k}}{\ln\frac{R}{s^+_{\delta,k}}}\frac{\partial\bar{G}(z_i^+,z_k^+)}{\partial z^+_{i,h}} \\
&+ \sum_{l=1}^n \frac{a^-_{\delta,l}}{\ln\frac{R}{s^-_{\delta,l}}}\frac{\partial\bar{G}(z_i^+,z_l^-)}{\partial z^+_{i,h}}\Big)\int_{B_1(0)} \phi^{p-1}(|x|)\phi'(|x|)\frac{x_h^2}{|x|} + 0\Big(\frac{\varepsilon^{2+\sigma}}{|\ln\varepsilon|^{p+1}}\Big) \\
=&\frac{4\pi^2\delta^2 a^+_{\delta,i}}{|\ln\varepsilon||\ln\frac{R}{s^+_{\delta,i}}|}\frac{\partial q(z_i^+)}{\partial z^+_{i,h}} + \frac{2\pi\delta^2(a^+_{\delta,i})^2}{(\ln\frac{R}{s^+_{\delta,i}})^2}\frac{\partial g(z_i^+,z_i^+)}{\partial z^+_{i,h}} - \sum_{k\neq i}^m \frac{2\pi\delta^2 a^+_{\delta,i}a^+_{\delta,k}}{|\ln\frac{R}{s^+_{\delta,k}}||\ln\frac{R}{s^+_{\delta,i}}|}\frac{\partial\bar{G}(z_i^+,z_k^+)}{\partial z^+_{i,h}} \\
&+ \sum_{l=1}^n \frac{2\pi\delta^2 a^+_{\delta,i}a^-_{\delta,l}}{|\ln\frac{R}{s^-_{\delta,l}}||\ln\frac{R}{s^+_{\delta,i}}|}\frac{\partial\bar{G}(z_i^+,z_l^-)}{\partial z^+_{i,h}} + O\Big(\frac{\varepsilon^{2+\sigma}}{|\ln\varepsilon|^{p+1}}\Big),
\end{aligned}
$$

since

$$\int_{B_1(0)} \phi^{p-1}(|x|)\phi'(|x|)\frac{x_h^2}{|x|} = -\frac{2\pi}{p}|\phi'(1)|.$$

Other terms can be estimated as above. Thus, the result follows.

□

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